Robust Control

Spring, 2019

Instructor: Prof. Masayuki Fujita (S5-303B)

1st class

Tue., 9th April, 2019, 10:45～12:15,

S423 Lecture Room

Reference:

[H95] R.A. Hyde,
H_∞ Aerospace Control Design: A VSTOL Flight Application,
Springer, 1995.

Harrier Jump Jet

Robust Control for

Flight Control

Process Control

Automotive Control

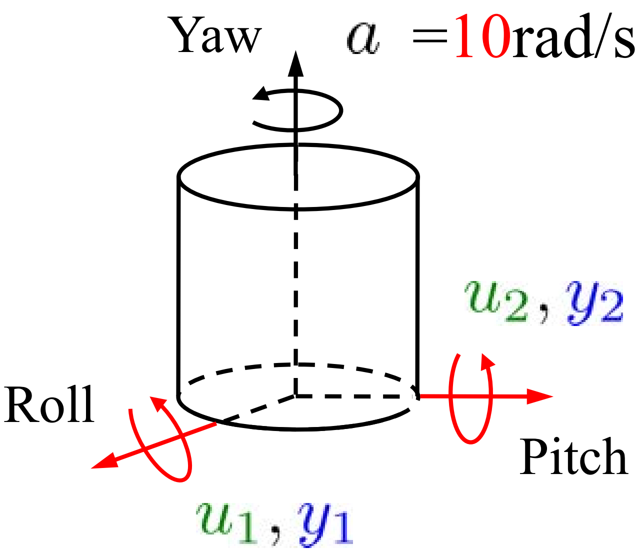
Mechatronics

Smart Grid

Motivating Example: Spinning Satellite's Attitude Control

JAXA: ETS-VIII

Spinning Satellite



Inputs: u_1 u_2 Torque

Outputs: y_1 y_2 Angular velocity

Multi-Input Multi-Output System
(MIMO System)

Single-Input Single-Output System
(SISO System)

Multivariable Plants

古典制御の時代が最初に壁にぶつかったのが「多変数」の問題である

[Tsien54] H. S. Tsien : *Engineering Cybernetics*, McGraw-Hill, 1954

[木村89] 木村 : 制御技術と制御理論，システム/制御/情報，33(6) 257/263, 1989

Spinning Satellite

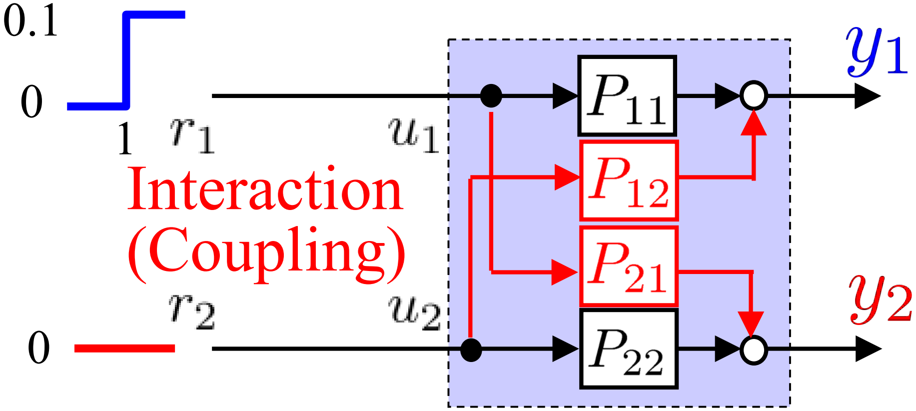
Transfer Function Matrix

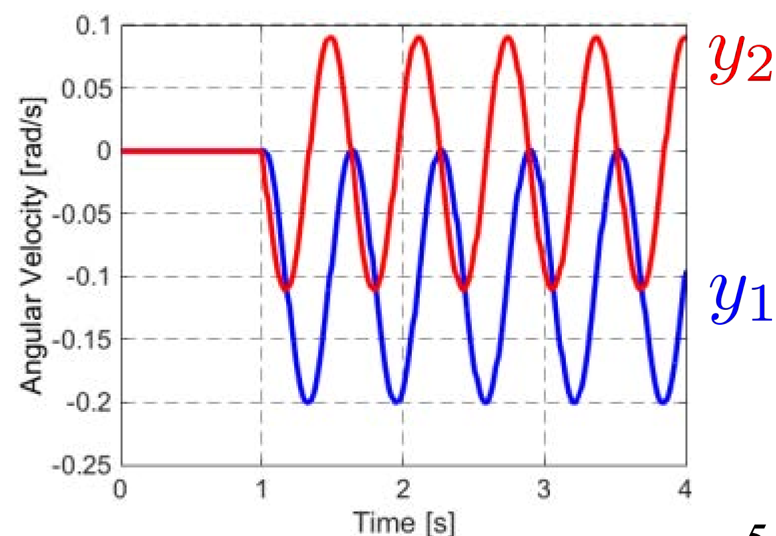
$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s-100}{s^2+100} & \frac{10s+10}{s^2+100} \\ \frac{-10s-10}{s^2+100} & \frac{s-100}{s^2+100} \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc} 0 & 10 & 1 & 0 \\ -10 & 0 & 0 & 1 \\ \hline 1 & 10 & 0 & 0 \\ -10 & 1 & 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \qquad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$



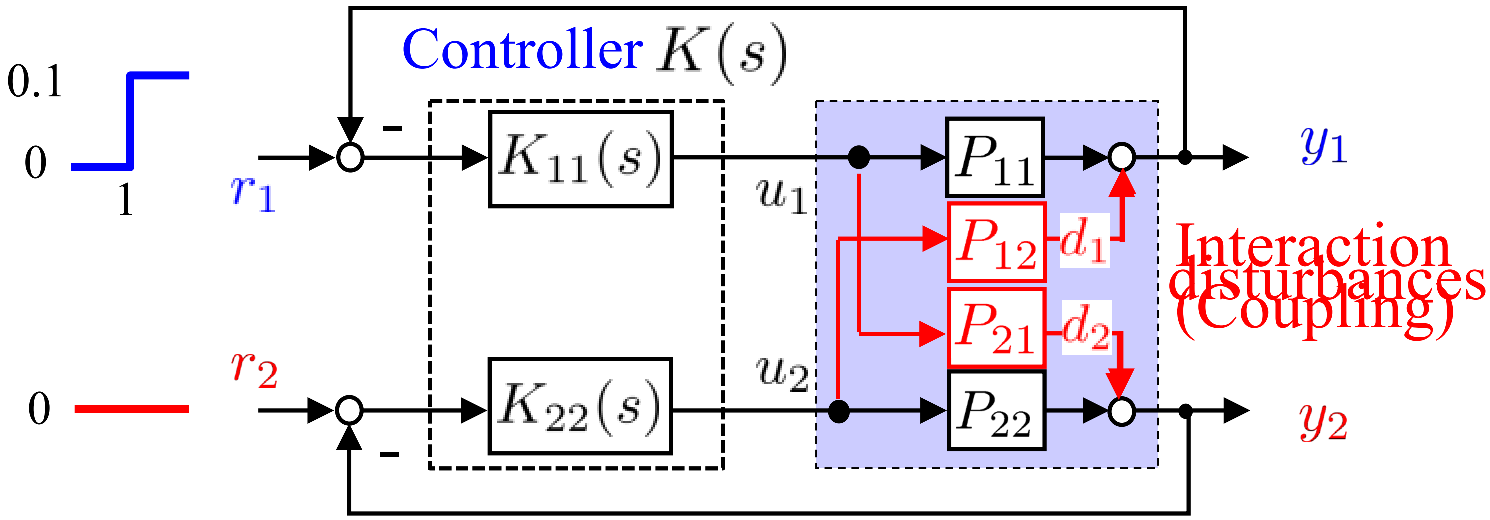


State Space Representation

Unified treatment for SISO/MIMO

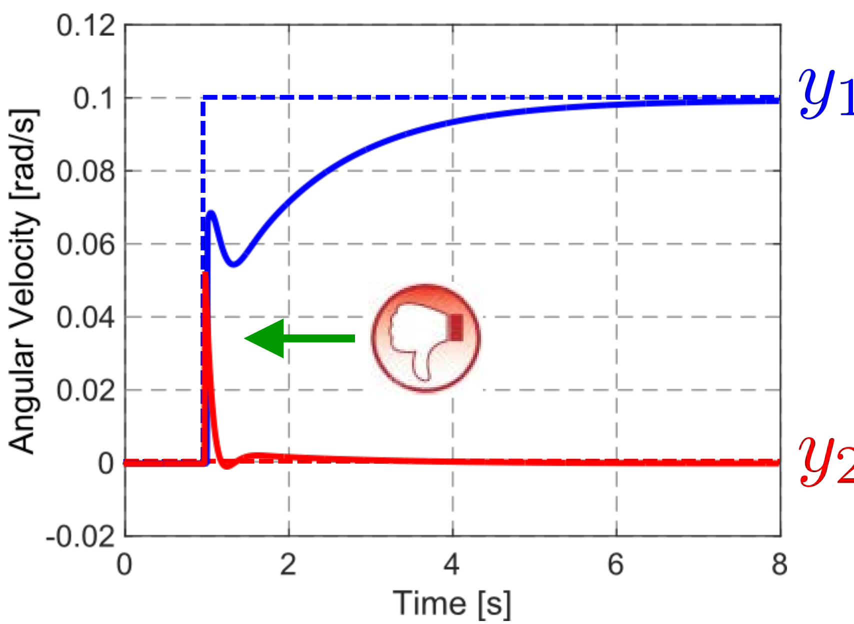
Control of Multivariable Plants [SP05, pp. 91-93]

1. Diagonal Controller (Decentralized Control)



$$K_{11}(s) = K_{22}(s) = -0.864 - \frac{1.34}{s} - 0.135s$$

$$K(s) = \begin{bmatrix} K_{11}(s) & 0 \\ 0 & K_{22}(s) \end{bmatrix}$$



MATLAB Command

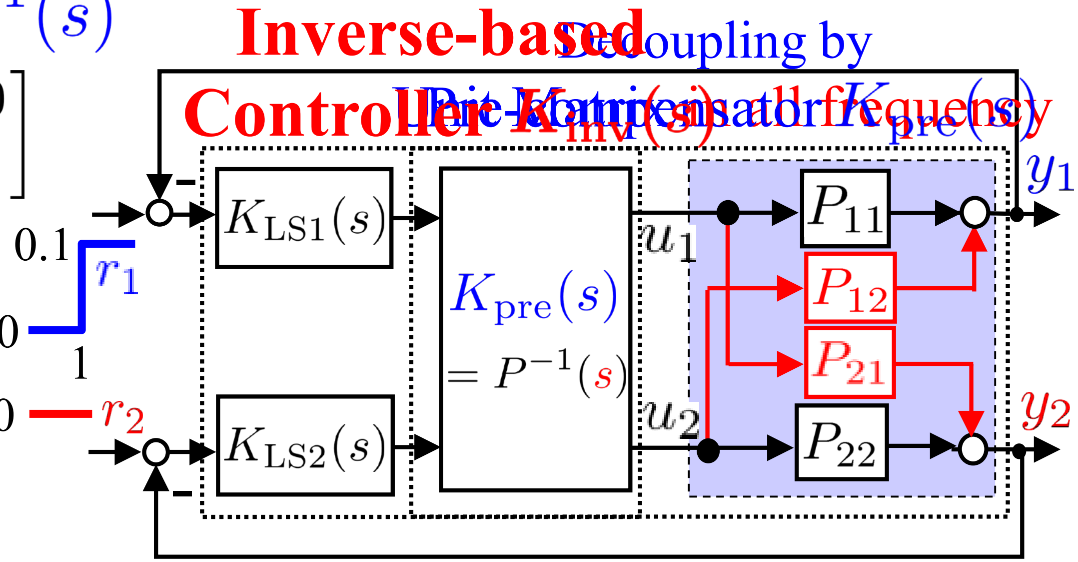
```
P11 = tf([1 -100],[1 0 100]) ;
K = pidtune( P11, 'PID') ;
```

Control of Multivariable Plants [SP05, pp. 91-93]

2. Dynamic Decoupling $P^{-1}(s)$

$$P^{-1}(s) = 0.01 \begin{bmatrix} s-100 & -10s-10 \\ 10s+10 & s-100 \end{bmatrix}$$

$$P'(s) = P(s)P^{-1}(s) = I$$

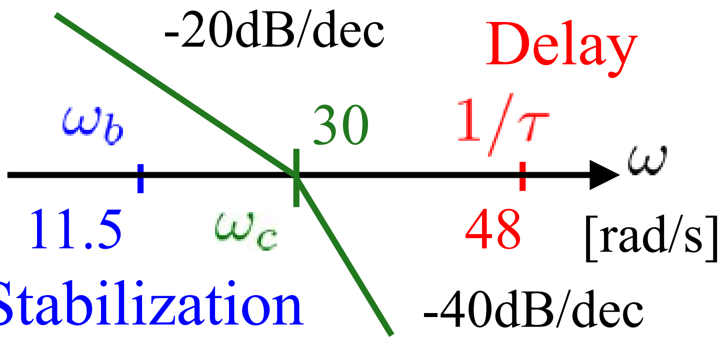


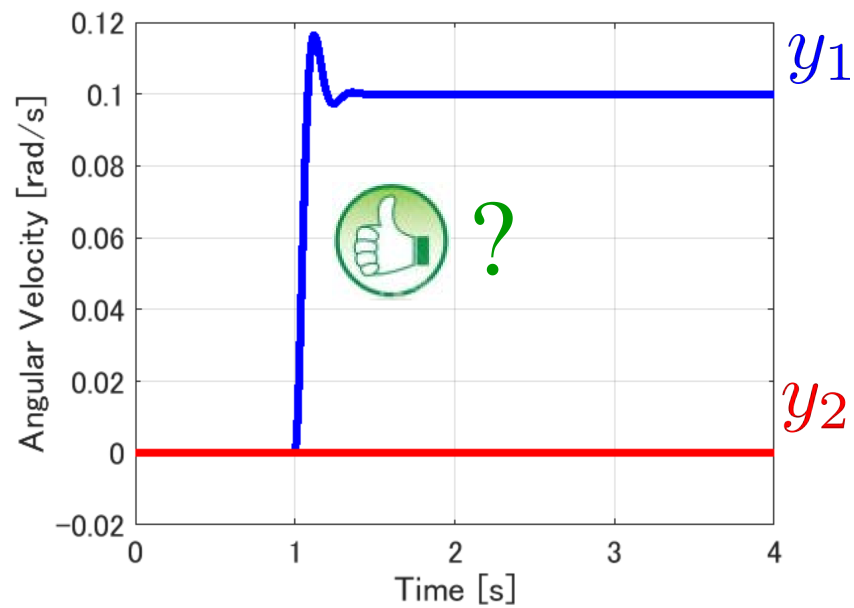
Loop Shaping Design

$$K_{\rm LS1}(s) = K_{\rm LS2}(s) = \frac{900}{s(s+30)}$$

Target Loop (Desired Loop)

$$L_{\rm target}(s) = \frac{900}{s(s+30)} I_2$$

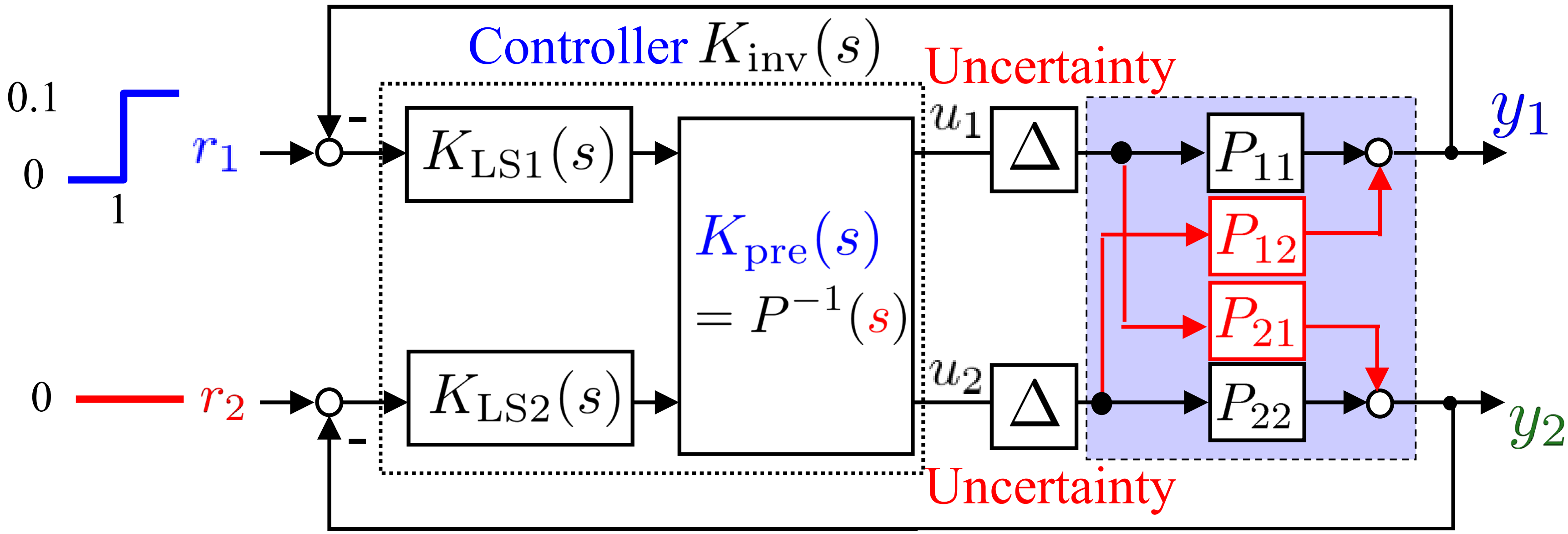




$$K_{\rm inv}(s) = P^{-1}(s) \begin{bmatrix} K_{\rm LS1}(s) & 0 \\ 0 & K_{\rm LS2}(s) \end{bmatrix}$$

Control of Multivariable Plants [SP05, pp. 91-93]

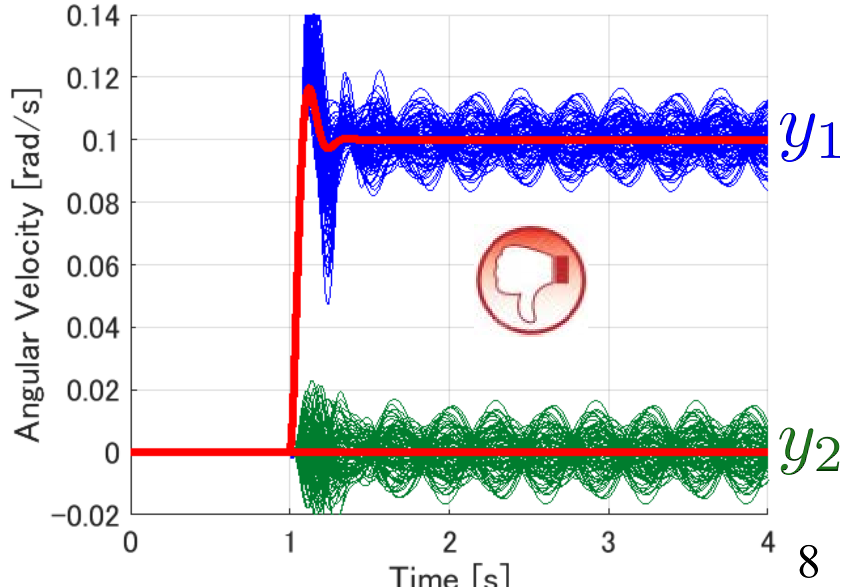
Inverse-based Controller $K_{\text{inv}}(s)$



$$P^{-1}(s) = 0.01 \begin{bmatrix} s-100 & -10s-10 \\ 10s+10 & s-100 \end{bmatrix}$$

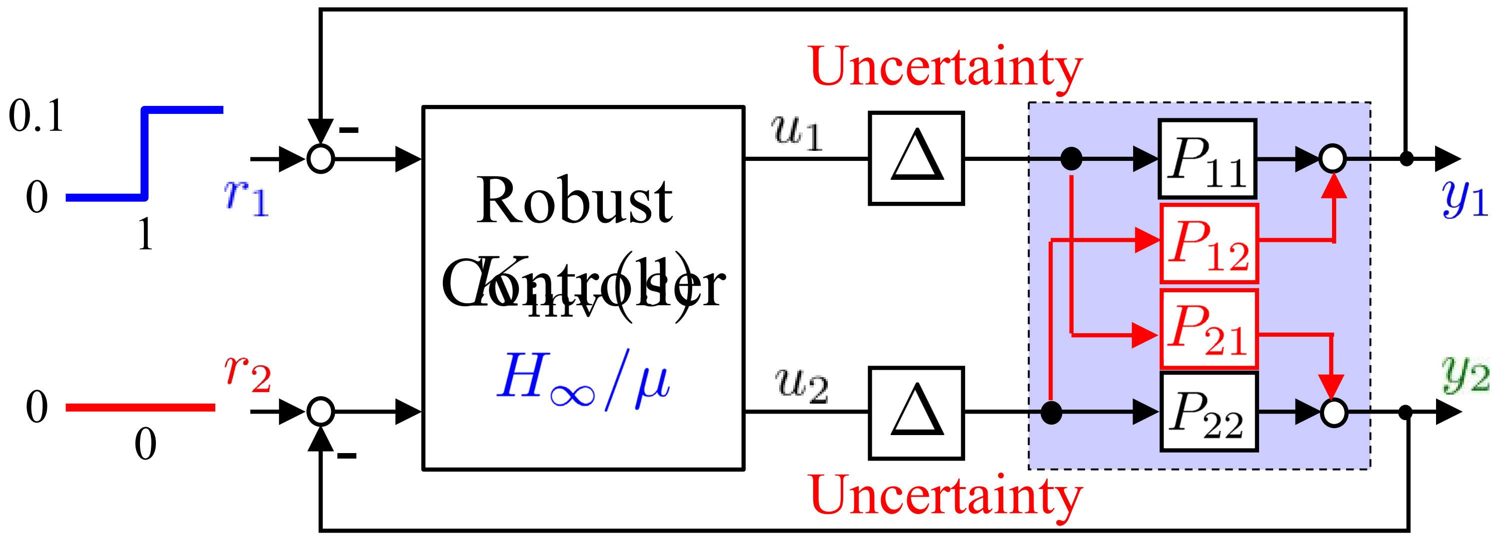
$$K_{\text{LS1}}(s) = K_{\text{LS2}}(s) = \frac{900}{s(s+30)}$$

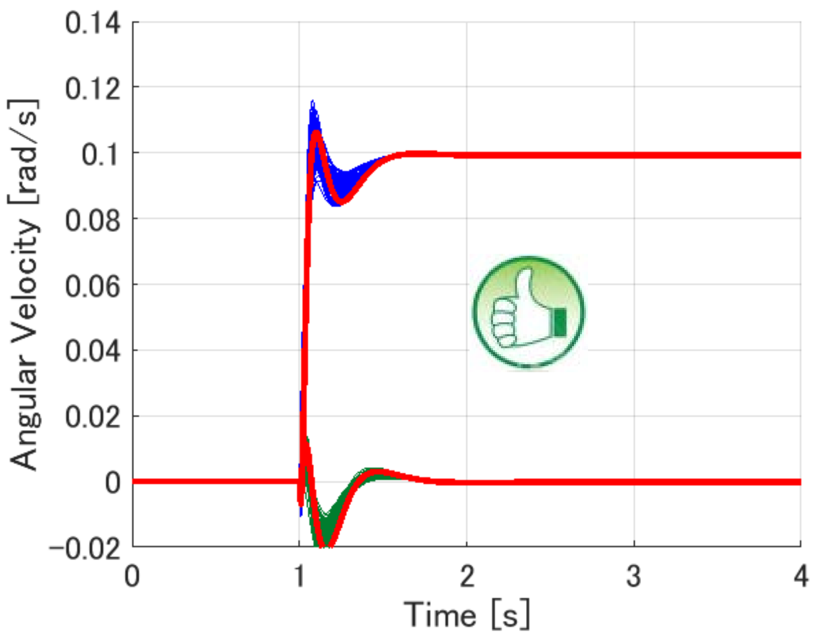
$$K_{\text{inv}}(s) = P^{-1}(s) \begin{bmatrix} K_{\text{LS1}}(s) & 0 \\ 0 & K_{\text{LS2}}(s) \end{bmatrix}$$

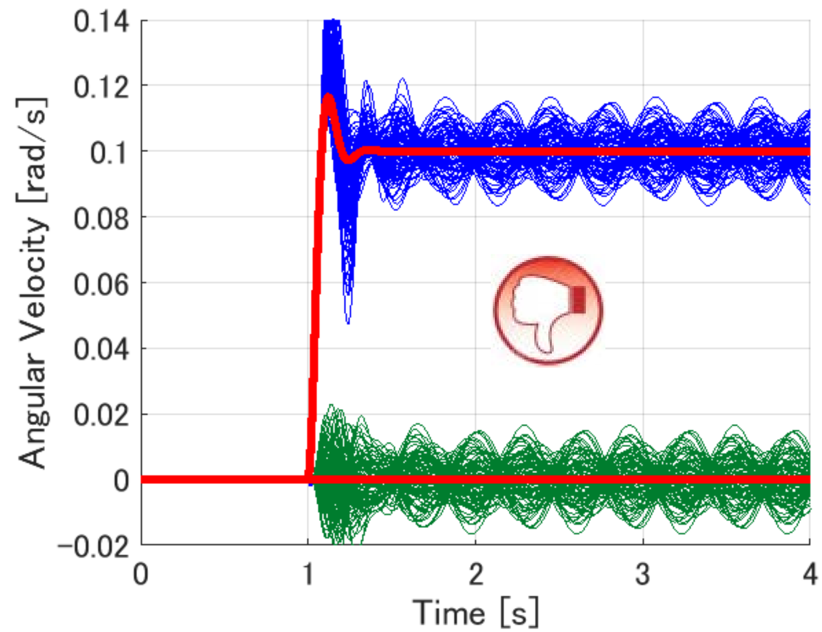


Control of Multivariable Plants

3. Robust Controller







Robust Control

Instructor: Prof. Masayuki Fujita (S5-303B)

Schedule: 9th, 16th, 23rd April, 7th, 14th, 21st, 28th May, 4th June

Units: 1 unit

Teaching Assistants (TA):

Hayato Dan, Made Widhi Surya Atman (S5-204A)

Reference:

[SP05] S. Skogestad and I. Postlethwaite,
***Multivariable Feedback Control; Analysis and Design, Second Edition*, Wiley, 2005.**

[ZD97] K. Zhou and J. C. Doyle,
***Essentials of Robust Control,* Prentice Hall, 1997.**

[M19] *Robust Control Toolbox Documentation(R2019a),* MathWorks, 2019.

Grading: Reports on 4th (45%) and 6th (55%) classes

(MATLAB: Robust Control Toolbox)

1. Multivariable Feedback Control and Nominal Stability

1.1 Multivariable Feedback Control

[SP05, Sec. 3.5]

1.2 Multivariable Frequency Response Analysis

[SP05, Sec. 3.3, A.3, A.5]

1.3 Internal Stability

[SP05, Sec. 4.1, 4.7]

1.4 All Stabilizing Controllers

[SP05, Sec. 4.8]

Reference:

[SP05] S. Skogestad and I. Postlethwaite, ***Multivariable Feedback Control; Analysis and Design,*** Second Edition, Wiley, 2005.

Frequency Response for SISO Systems

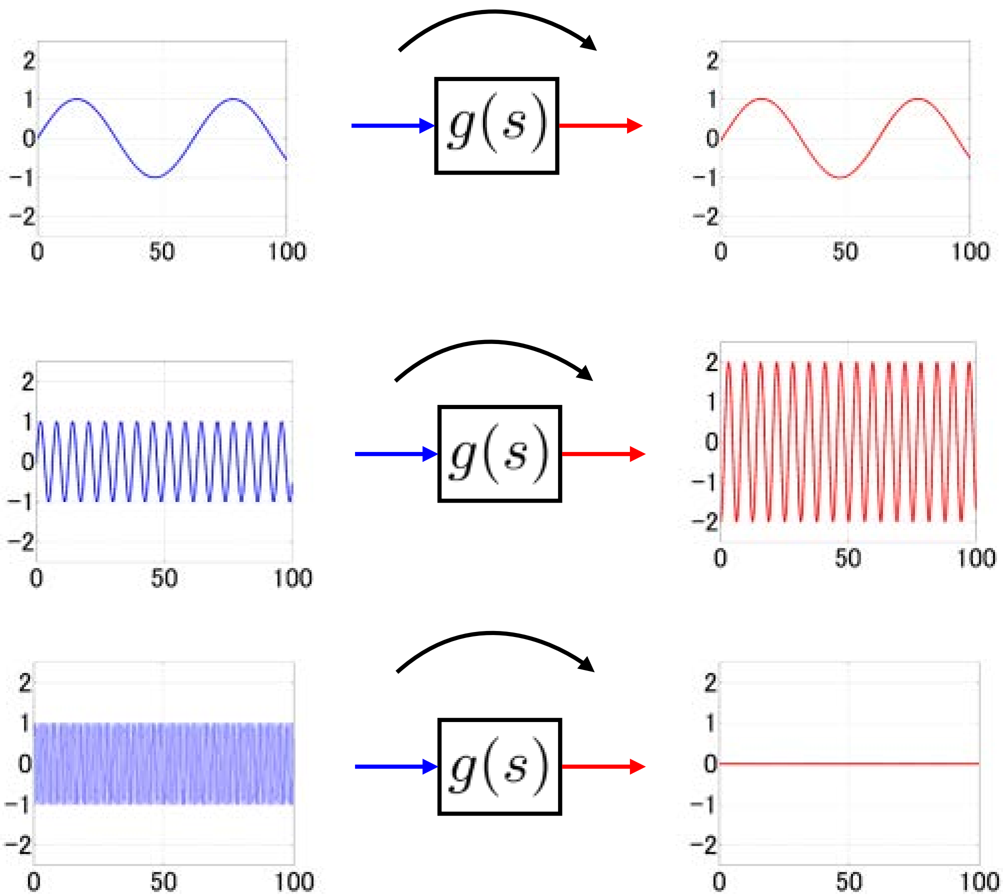
[Ex.] $$g(s) = \frac{1}{s^2 + 0.5s + 1} \quad \omega_n = 1 \quad \zeta = 0.25$$

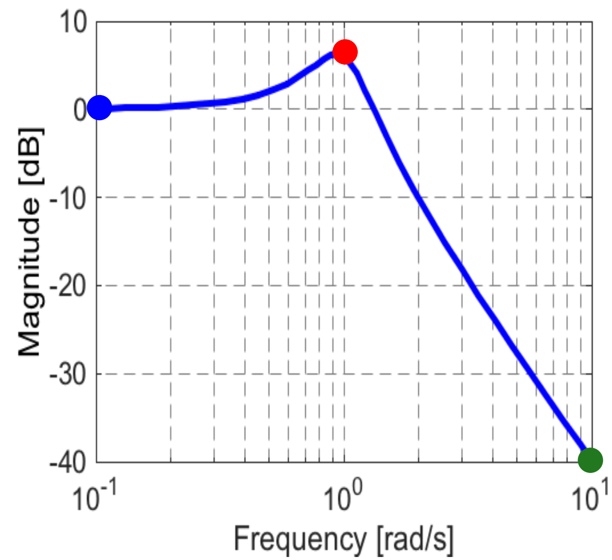
$u(t) = \sin(\omega t)$ $\qquad$ $y(t) = |g(j\omega)| \sin(\omega t + \angle g(j\omega))$

$\omega = 0.1$

$\omega = 1$

$\omega = 10$





Bode Plot
(Gain)

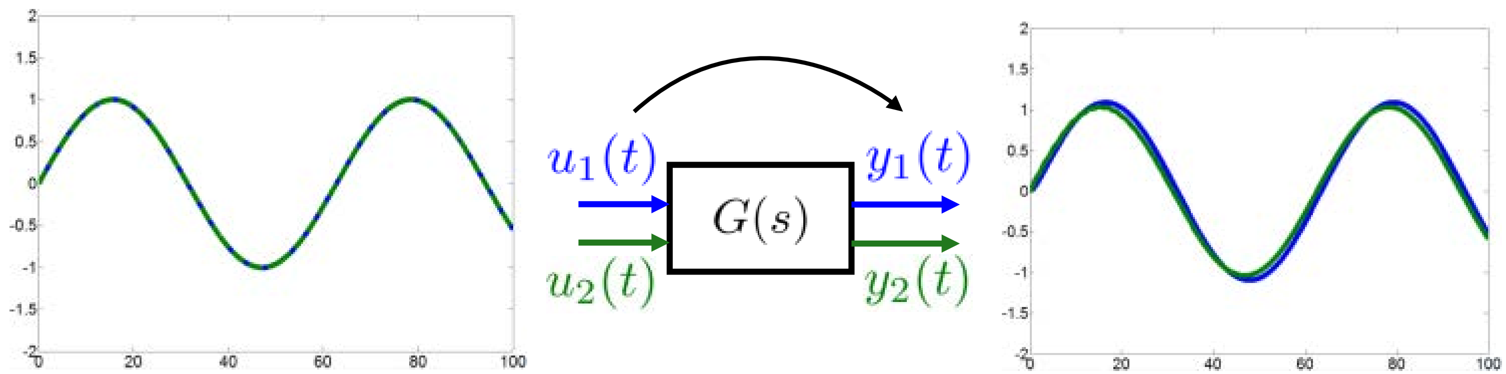
Frequency Response for MIMO Systems

[Ex.]

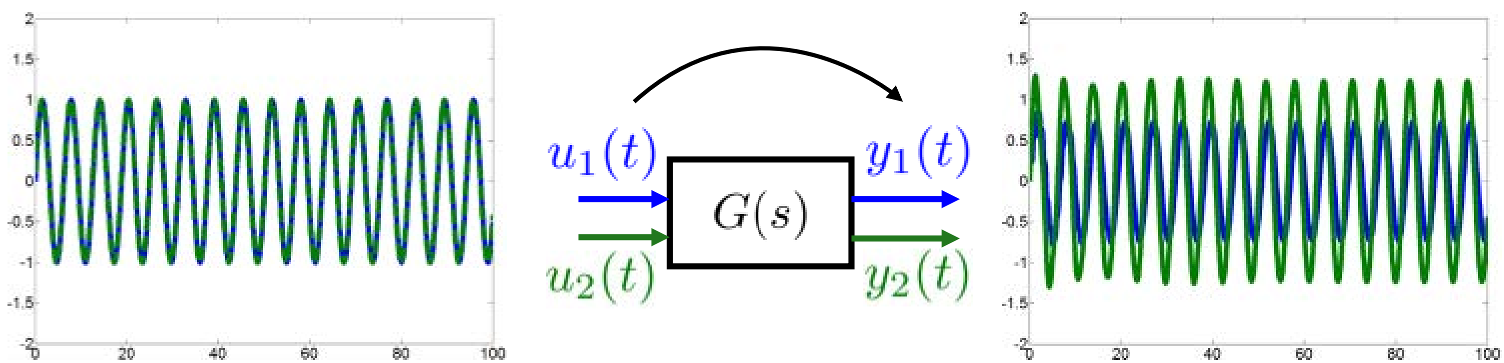
$$G(s) = \begin{bmatrix} \frac{10(s+1)}{s^2+0.2s+100} & \frac{1}{s+1} \\ \frac{s+2}{s^2+0.1s+10} & \frac{5(s+1)}{(s+2)(s+3)} \end{bmatrix}$$

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \qquad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

$\omega = 0.1$



$\omega = 1$



SISO ➡ MIMO ?

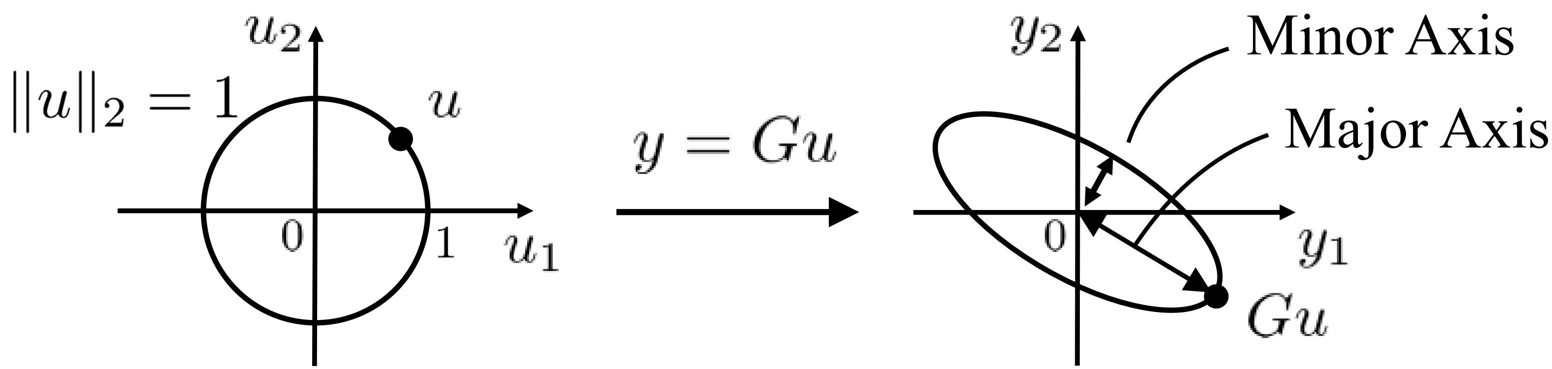
Singular Value Decomposition [SP05, A.3]

A. J. Laub

[SP05, Ex. 3.3] (p. 74)

svd(G)

$$G = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \Rightarrow G = \begin{bmatrix} -0.87 & -0.48 \\ -0.48 & 0.87 \end{bmatrix} \begin{bmatrix} \underline{7.34} & 0 \\ 0 & \underline{0.27} \end{bmatrix} \begin{bmatrix} -0.79 & -0.60 \\ -0.60 & -0.79 \end{bmatrix}$$



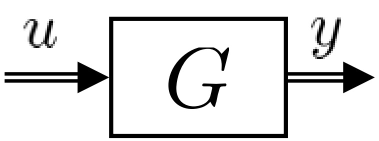
$$G = U \Sigma V^H$$

U, V : Unitary Matrices

Singular Values $\sigma_1, \cdots, \sigma_p$

$(\sigma_1 > \sigma_2 > \cdots > \sigma_p)$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \sigma_p & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



$y = Gu$

$\sigma_i = \sqrt{\lambda_i(G^H G)}$ λ_i : i -th eigenvalue

Maximum Singular Value

$$\bar{\sigma}(G) = \sigma_1 = \max_{u \neq 0} \|y\|_2 / \|u\|_2$$

Minimum Singular Value

$$\underline{\sigma}(G) = \sigma_p = \min_{u \neq 0} \|y\|_2 / \|u\|_2$$

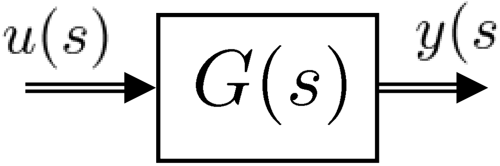
σ -plot [SP05, p. 79]

SISO: Absolute value $|g(j\omega)|$

MIMO: Singular value plot $\sigma(G(j\omega))$

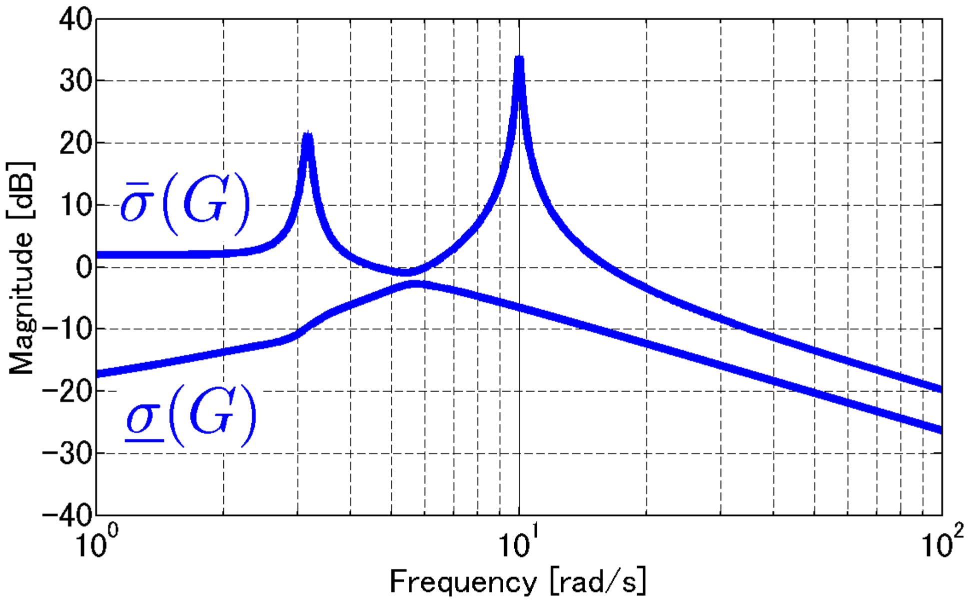
[Ex.]

$$G(s) = \begin{bmatrix} \frac{10(s+1)}{s^2+0.2s+100} & \frac{1}{s+1} \\ \frac{s+2}{s^2+0.1s+10} & \frac{5(s+1)}{(s+2)(s+3)} \end{bmatrix}$$



$$y(\omega) = G(j\omega)u(\omega)$$

σ -plot of G



Extension of Bode gain plot to MIMO Systems

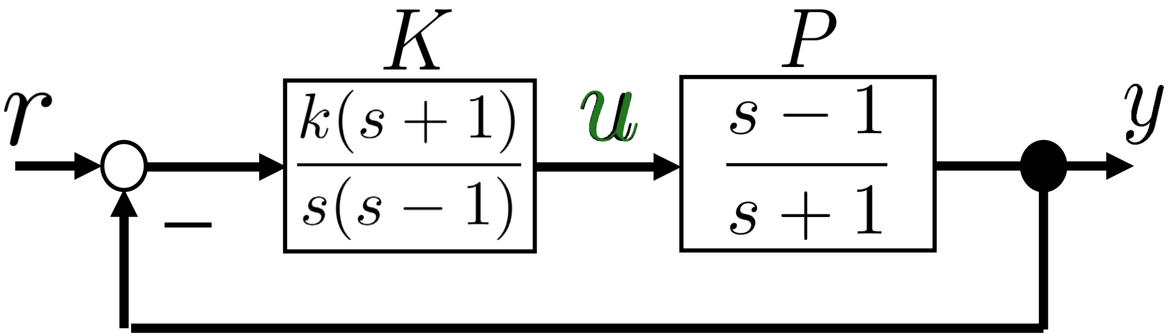
MATLAB Command

```
num = { [10 10], 1;
            [1 2],   [5 5] };
den = { [1 0.2 100], [1 1];
            [1 0.1 10],   [1 5 6] };
G = tf( num, den );

figure
sigma(G)
```

Motivating Example for Internal Stability in SISO Systems

[SP05, Ex. 4.16] (p. 144)



C.A. Desoer and W.S. Chan, Journal of the Franklin Institute, 300 (5-6) 335-351, 1975

C.A. Desoer

*The Feedback Interconnection of Lumped Linear Time-invariant Systems**

by C. A. DESOER *and* W. S. CHAN

Department of Electrical Engineering and Computer Sciences and the Electronics Research Laboratory, University of California, Berkeley, California

ABSTRACT: *This paper considers the feedback interconnection of two multi-input multi-output subsystems characterized by* rational *transfer functions* $\hat{G}_1$ *and* $\hat{G}_2$. *These transfer functions are* not *assumed to be* proper *nor* exponentially stable. *The effect of output disturbances on stability is taken into account. Ten examples are given to show that instabilities may appear anywhere around the loop. Next, under a sequence of successively*

Closed Loop Transfer Function $(r \to y)$

$$G_{yr} = \frac{k}{s+k} \qquad (k > 0)$$

Stable?

Another Closed Loop Transfer Function $(r \to u)$

$$G_{ur} = \frac{k(s+1)}{(s-1)(s+k)}$$

Unstable!!

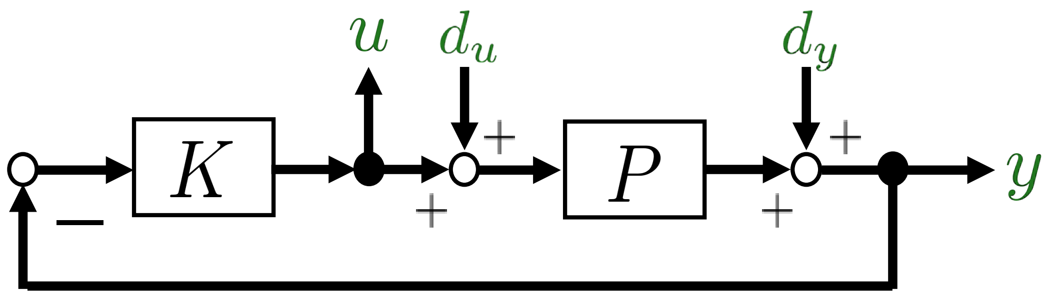
Why?

$$L = PK = \frac{s-1}{s+1}\,\frac{k(s+1)}{s(s-1)} = \frac{k}{s}$$

Unstable Pole/Zero Cancellation

Gang of Four (SISO)

In order to avoid pole/zero cancellation, consider input injection & output measurement for each dynamic block.



[AM08]
K. J. Astrom and R. Murray, *Feedback Systems*, Princeton University Press, 2008

Sensitivity

$$\underset{(d_y \to y)}{S} = \frac{1}{1+PK}$$

Complementary Sensitivity

$$\underset{(d_u \to u)}{T} = \frac{PK}{1+PK}$$

Load Sensitivity

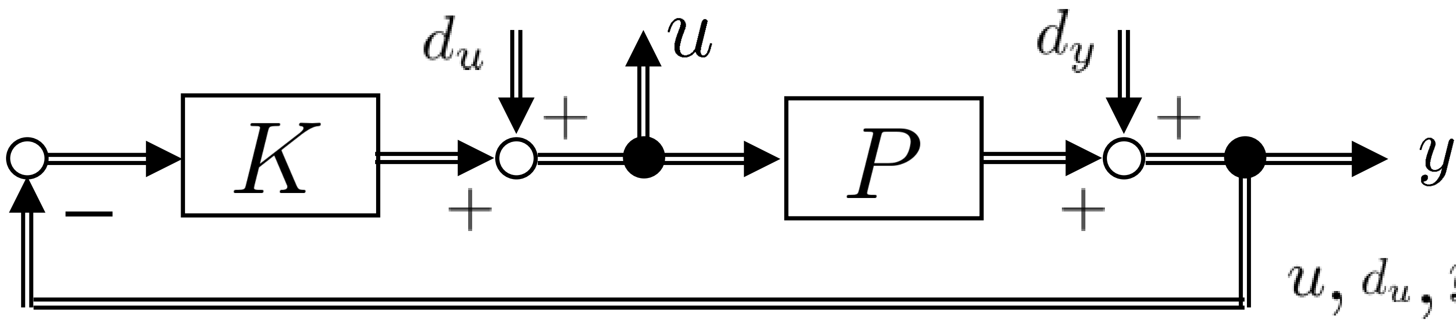
$$\underset{(d_u \to y)}{PS} = \frac{P}{1+PK}$$

Noise Sensitivity

$$\underset{(d_y \to u)}{KS} = \frac{K}{1+PK}$$

Internal Stability of Multivariable Feedback Systems

Nominal Stability [SP05, Fig. 4.3] (p. 145)



u, d_u, y, d_y : Vectors

$P, \ K$: Transfer function matrices

Well-posedness: $1 + P(\infty)K(\infty) \neq 0$

(Gang of Four: well-defined and proper)

$$u = (I + KP)^{-1} d_u - K(I + PK)^{-1} d_y$$

$$y = P(I + KP)^{-1} d_u + (I + PK)^{-1} d_y$$

[SP05, Theorem 4.6] (p. 145) Nominal Stability(NS) Test

Assume $P, \ K$ contain no unstable hidden modes.

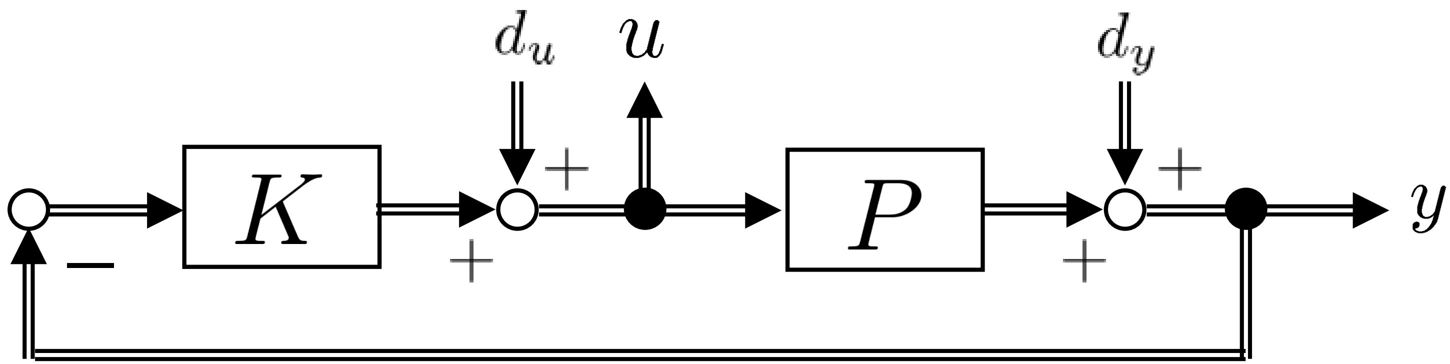
Then, the feedback system in the figure is ***internally stable***

if and only if all four closed-loop transfer matrices are stable.

Internal Stability of Multivariable Feedback Systems

Nominal Stability

[SP05, Fig. 4.3] (p. 145)



State-space representation: $\left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$ [SP05, p. 124]

[ZD97, Theorem 5.5](p. 70) Nominal Stability(NS) Test

The system is internally stable iff $\bar{A}$ is stable

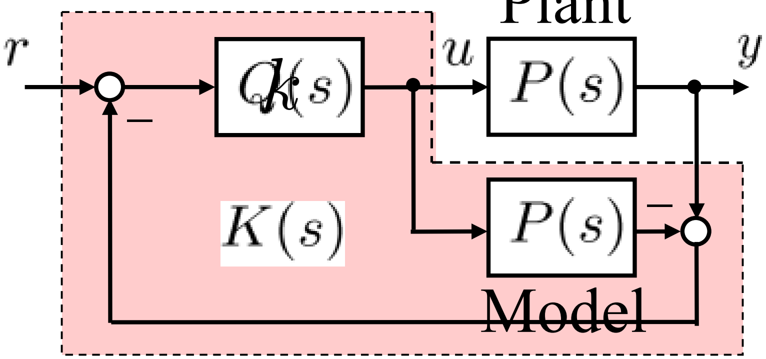
[ZD97] K. Zhou and J. C. Doyle, *Essentials of Robust* control, Prentice Hall, 1997

Youla-parameterization (Q-parameterization)
Internal Model Control (IMC) Structure

[SP05, p. 148]

Stable Plant $P(s) \in \mathcal{S}$: Proper Stable Transfer Function Matrices

D.C.Youla



$Q(s) \in \mathcal{S}$

All Stabilizing Controllers $K(s) = (I - QP)^{-1}Q$

$$= Q(I - PQ)^{-1}$$

Gang of Four

$$(I + PK)^{-1} = \underline{I - PQ} \qquad (I + KP)^{-1} = \underline{I - QP}$$

$$P(I + KP)^{-1} = \underline{P(I - QP)} \qquad K(I + PK)^{-1} = \underline{Q}$$

Surprising Fact: Necessary and Sufficient

$Q(s) = K(I + PK)^{-1} \in \mathcal{S}$ ⇒ Internally stable

Internally stable ⇒ $Q(s) = K(I + PK)^{-1} \in \mathcal{S}$

Youla-Kucera-parameterization

Unstable Plant $P(s)$ [SP05, p. 149]

Left Coprime Factorization [SP05, p. 122]
(can be also on the right)

M. Vidyasagar,
The MIT Press,1985

$$P(s) = M_l(s)^{-1} N_l(s) \quad M_l(s), N_l(s) \in \mathcal{S}$$

Coprime: No common unstable zeros

$^{\exists} X_r(s), Y_r(s) \in \mathcal{S}$ s.t. $N_l X_r + M_l Y_r = I$ (Bezout Identity)

$\xleftrightarrow{\text{iff}}$ $M_l(s), N_l(s)$: Stable coprime transfer funcion matrices

All Stabilizing Controllers

$$K(s) = (Y_r - QN_l)^{-1}(X_r + QM_l) \quad Q(s) \in \mathcal{S}$$

Q : Stable transfer function matrix satisfying

$$\det(Y_r(\infty) - Q(\infty)N_l(\infty)) \neq 0$$

Youla-Kucera-parameterization (Unstable Plants)

[SP05, p. 149]

[SP05, Ex. 4.1]

$$P(s) = \frac{(s-1)(s+2)}{(s-3)(s+4)} = \frac{\left(\frac{s-1}{s+4}\right)}{\left(\frac{s-3}{s+2}\right)}, \quad N(s) = \frac{s-1}{s+4}, \quad M(s) = \frac{s-3}{s+2} \quad (*)$$

$$M(s), N(s) : (*) \Rightarrow X(s) = \frac{s+32}{2s+4}, \quad Y(s) = \frac{s-16}{2s+8} \quad X(s), Y(s) \in \mathcal{S}$$

$$NX + MY = 1$$ Bezout Identity

$$\Rightarrow \frac{X(s)}{Y(s)} = \frac{s^2+36s+128}{s^2-14s-32} = K(s)$$ A Stabilizing Controller! $(Q(s) = 0)$

$$\Rightarrow \frac{X(s)+Q(s)M(s)}{Y(s)-Q(s)N(s)} = K(s)$$ All Stabilizing Controllers! $(Q(s) \in \mathcal{S})$

Stable Plant Case

$$N = P, M = 1, X = 0, Y = 1 \Rightarrow K(s) = (I - QP)^{-1}Q$$

State-Space Computation of All Stabilizing Controllers

State Space Representation [SP05, p. 124]

$$P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = M_l^{-1} N_l$$

All Stabilizing Controllers

$$K(s) = (Y_r - QN_l)^{-1}(X_r + QM_l) \qquad Q(s) \in \mathcal{S}$$

Let matrices F, H be such that $A + BF$, $A + HC$ are stable

$$\begin{bmatrix} X_r & Y_r \\ N_l & M_l \end{bmatrix} = \left[\begin{array}{c|cc} A + HC & -B - HD & H \\ \hline F & I & 0 \\ C & -D & I \end{array}\right]$$

Matrix Computation

System Structure on Controllers

If $Q = 0$, then K is

State Feedback + Observer

$$\begin{cases} u = F\hat{x} \\ \dot{\hat{x}} = A\hat{x} + Bu + H(y - C\hat{x}) \end{cases}$$

Completion of Linear Feedback System Theory

A stabilizing controller

State feedback/Observer

All stabilizing controllers

Q (Youla) Parametrization

Transfer Function

Pole/Zero

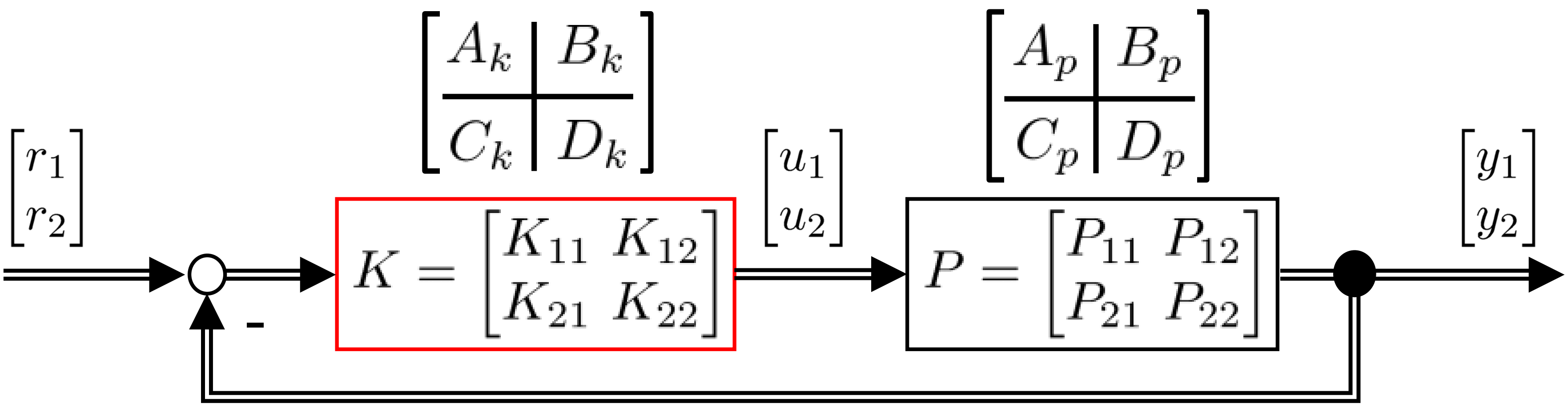
Structure

Controllability,
Observability

State Space Form

(Data Structure)

State x



1. Multivariable Feedback Control and Nominal Stability

1.1 Multivariable Feedback Control

[SP05, Sec. 3.5]

1.2 Multivariable Frequency Response Analysis

[SP05, Sec. 3.3, A.3, A.5]

1.3 Internal Stability

[SP05, Sec. 4.1, 4.7]

1.4 All Stabilizing Controllers

[SP05, Sec. 4.8]

Reference:

[SP05] S. Skogestad and I. Postlethwaite, ***Multivariable Feedback Control; Analysis and Design,*** Second Edition, Wiley, 2005.

2. Nominal Performance

2.1 Weighted Sensitivity [SP05, Sec. 2.8, 3.3, 4.10, 6.2, 6.3]

2.2 Nominal Performance [SP05, Sec. 2.8, 3.2, 3.3]

2.3 Sensitivity Minimization [SP05, Sec. 3.2, 3.3, 9.3]

2.4 Remarks on Fundamental Limitations [SP05, Sec. 6.2]

Reference:

[SP05] S. Skogestad and I. Postlethwaite, ***Multivariable Feedback Control; Analysis and Design,*** Second Edition, Wiley, 2005.

Relative Gain Array [SP05, Sec. 3.4]

[SP05, Ex. 3.9] (pp. 85)

$$y_1 = u_1 + u_2$$

$$y_2 = 0.4u_1 - 0.1u_2$$

Transfer Function Matrix

$$G = \begin{bmatrix} 1 & 1 \\ 0.4 & -0.1 \end{bmatrix}$$

Relative Gain Array

$$\mathrm{RGA}(G) = \Lambda(G) := G \times (G^{-1})^T = \begin{bmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{bmatrix}$$

element wise multiplication

Pairing rule 1 Prefer paring on RGA elements close to 1

Use u_2 to control y_1 and use u_1 to control y_2

Pairing rule 2 Avoid pairing on negative RGA elements

Pairing rule 2 is satisfied for this choice

$$y_1 = u_1 + u_2$$

$$y_2 = -0.1u_1 + 0.4u_2$$

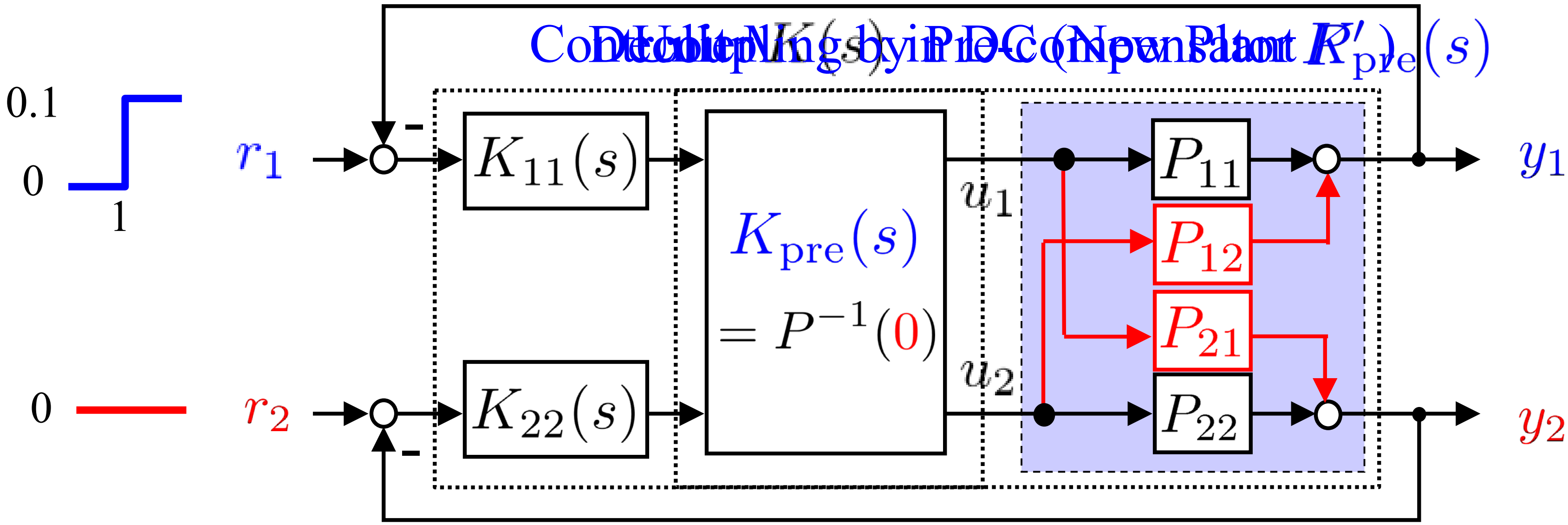
$$\Lambda = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

Rule 1

Rule 2

Control of Multivariable Plants [SP05, pp. 91-93]

Steady-State Decoupling $P^{-1}(0)$

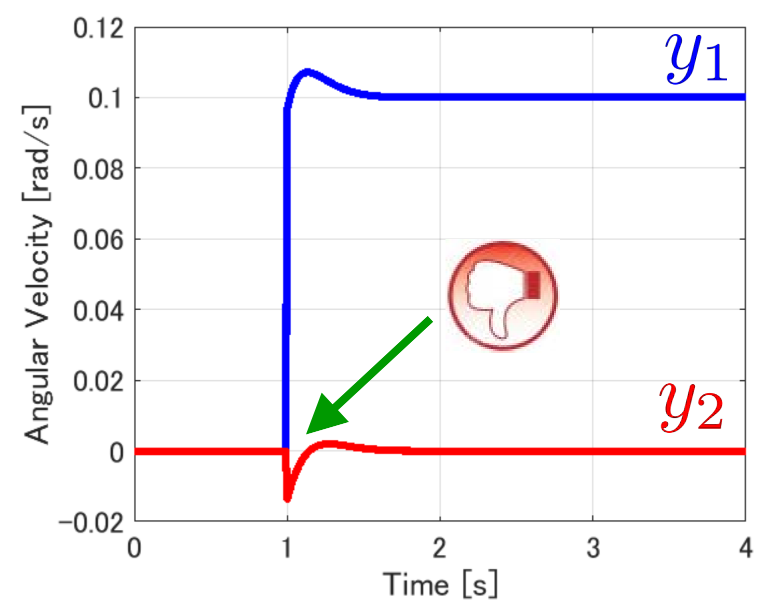


$$P'(s) = P(s)P^{-1}(0) = P(s)\begin{bmatrix} -1.0 & -0.1 \\ 0.1 & -1.0 \end{bmatrix}$$

$$P'(0) = I$$

$$K_{11}(s) = K_{22}(s) = 11.7 + \frac{49.1}{s} + 0.7s$$

$$K(s) = P^{-1}(0)\begin{bmatrix} K_{11}(s) & 0 \\ 0 & K_{22}(s) \end{bmatrix}$$



Poles [SP05, 4.4]

[SP05, Theorem 4.4] (p. 135)

The pole polynomial $\phi(s)$ corresponding to a minimal realization of a system with transfer function $G(s)$ is the least common denominator of all non-identically zero minors of all orders of $G(s)$.

[SP05, Ex. 4.10] (pp. 136, 139)

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 & \frac{s-1}{(s+1)(s+2)} \\ \frac{-1}{s-1} & \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix}$$

The minors of order 1

$$M^{2}_{23} = \det \begin{bmatrix} \frac{1}{s+1} & 0 & \frac{s-1}{(s+1)(s+2)} \\ \frac{-1}{s-1} & \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix} = \frac{1}{s+1}$$

$$M^{2}_{12} = \frac{s-1}{(s+1)(s+2)} \qquad M^{1}_{23} = \frac{-1}{s-1}$$

$$M^{1}_{13} = \frac{1}{s+2} \qquad M^{1}_{12} = \frac{1}{s+2}$$

The minors of order 2

$$M_2 = \det \begin{bmatrix} \frac{1}{s+1} & 0 & \frac{s-1}{(s+1)(s+2)} \\ \frac{-1}{s-1} & \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix} = \frac{2}{(s+1)(s+2)}$$

$$M_1 = \frac{-(s-1)}{(s+1)(s+2)^2}$$

$$M_3 = \frac{1}{(s+1)(s+2)}$$

The least common denominator of all the minors

$$\phi(s) = (s+1)(s+2)^2(s-1)$$

→ Poles $p = 1, -1, -2, -2$

Zeros [SP05, Sec. 4.5]

[SP05, Theorem 4.5] (p. 139)

The zero polynomial $z(s)$, corresponding to a minimal realization of the system, is the greatest common divisor of all the numerators of all order-r minors of $G(s)$, where r is the normal rank of $G(s)$, provided that these minors have been adjusted in such a way as to have the pole polynomial $\phi(s)$ as their denominator.

[SP05, Ex. 4.10] (pp. 136, 139) (Cont.)

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 & \frac{s-1}{(s+1)(s+2)} \\ \frac{-1}{s-1} & \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix}$$

Normal rank: 2

$$\phi(s) = (s+1)(s+2)^2(s-1)$$

The minors of order 2

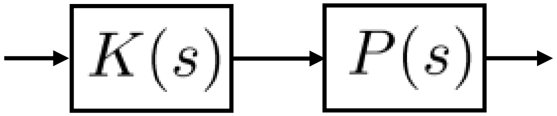
$$M_1 = \frac{-(s-1)}{(s+1)(s+2)^2} = \frac{-(s-1)^2}{\phi(s)} \qquad M_2 = \frac{2}{(s+1)(s+2)} = \frac{2(s-1)(s+2)}{\phi(s)}$$

$$M_3 = \frac{1}{(s+1)(s+2)} = \frac{(s-1)(s+2)}{\phi(s)}$$

The greatest common divisor of numerator

$$z(s) = (s-1)$$ → Zeros $z = 1$

Pole/Zero Cancellation [SP05, Sec. 4.5]



$$P(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{s+1} & \frac{1}{s+4} \end{bmatrix} \quad \text{Poles } p = -1, -4$$

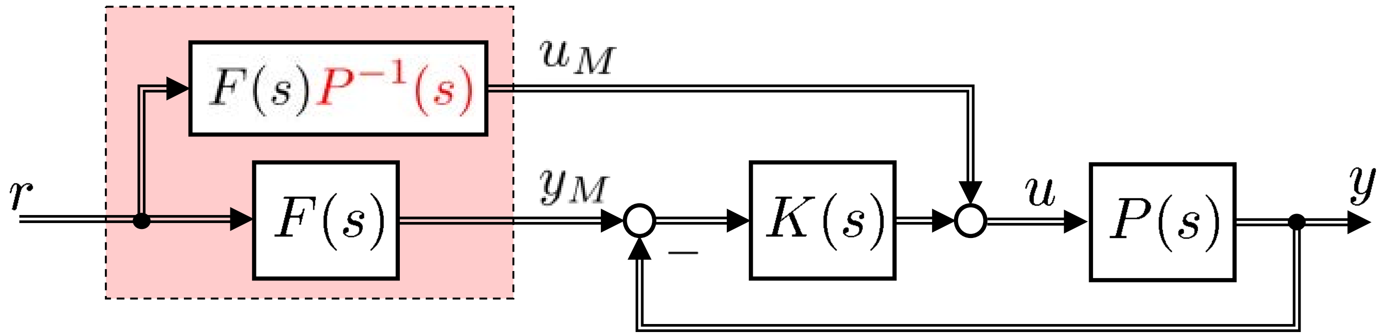
$$K(s) = \begin{bmatrix} 1 & 0 \\ -\frac{s+4}{s+1} & \frac{1}{s+2} \end{bmatrix} \quad \text{Poles } p = -1, -2$$

➡ Poles of $P(s)$ and $K(s)$: $\underline{-1, -1, -2, -4}$

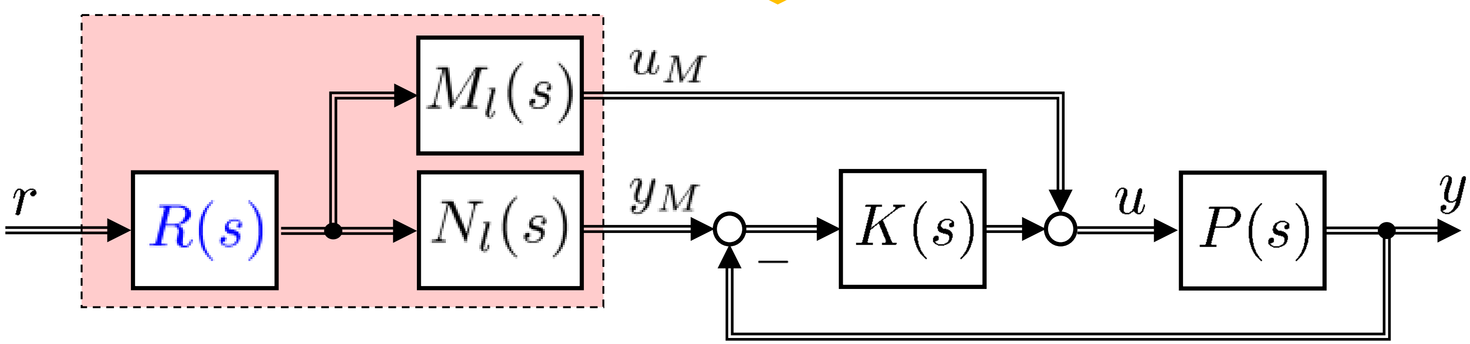
$$L(s) = PK = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{(s+2)(s+4)} \end{bmatrix} \quad \text{Poles } p = \underline{-1, -2, -4}$$

Poles of $P(s)$, $p = -1$ is cancelled

Two degrees of freedom Controller [SP05, p. 147]



$$y = F(s)r$$



Parameterize $F(s) = N_l(s)R(s)$ $\quad R(s)$: Stable matrix

$$P(s) = M_l^{-1}N_l \quad N_l = \left[\begin{array}{c|c} A+HC & -B-HD \\ \hline C & -D \end{array}\right]$$

$$M_l = \left[\begin{array}{c|c} A+HC & H \\ \hline C & I \end{array}\right]$$