Robust Control

Spring, 2019

Instructor: Prof. Masayuki Fujita (S5-303B)

2nd class

Tue., 16th April, 2019, 10:45～12:15,

S423 Lecture Room

2. Nominal Performance

2.1 Weighted Sensitivity [SP05, Sec. 2.8, 3.3, 4.10, 6.2, 6.3]

2.2 Nominal Performance [SP05, Sec. 2.8, 3.2, 3.3]

2.3 Sensitivity Minimization [SP05, Sec. 3.2, 3.3, 9.3]

2.4 Remarks on Fundamental Limitations [SP05, Sec. 6.2]

Reference:

[SP05] S. Skogestad and I. Postlethwaite,
Multivariable Feedback Control; Analysis and Design,
Second Edition, Wiley, 2005.

Sensitivity as Feedback Performance in SISO Systems

Disturbance Attenuation

Open-loop / Closed-loop

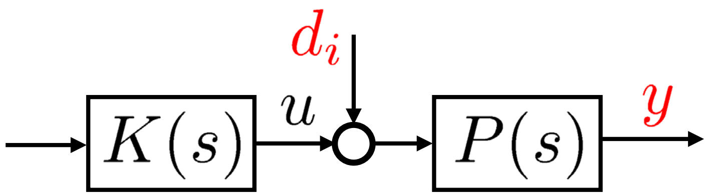
$d_i \to y$

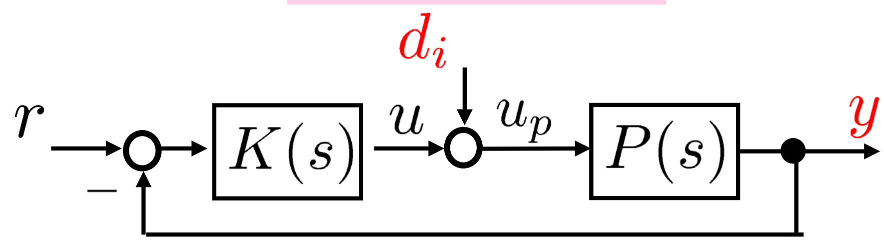
Open-loop:

$$y(s) = P(s)d_i(s)$$

Closed-loop:

$$y(s) = \frac{1}{1+P(s)K(s)}P(s)d_i(s)$$





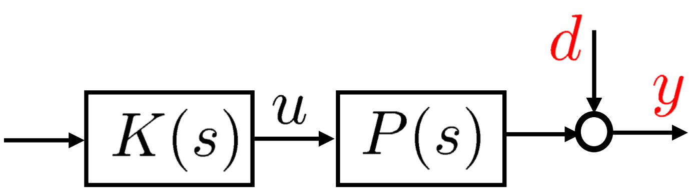
$d \to y$

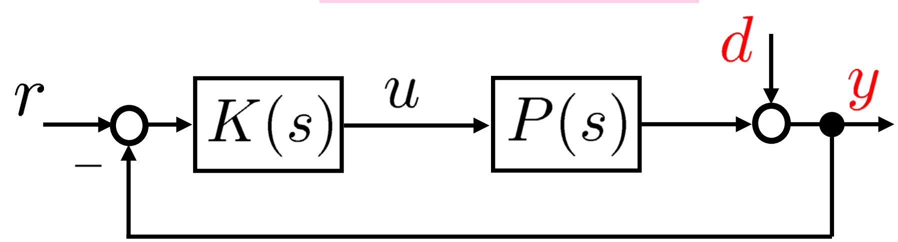
Open-loop:

$$y(s) = d(s)$$

Closed-loop:

$$y(s) = \frac{1}{1+P(s)K(s)}d(s)$$



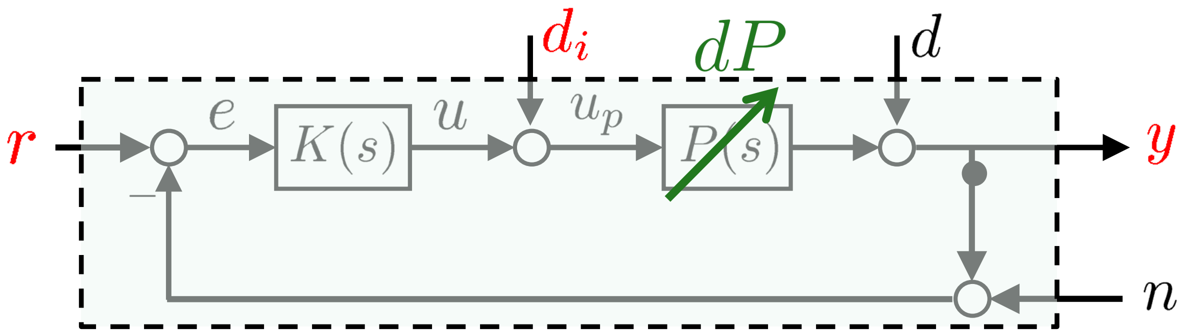


$$S(s) = \frac{1}{1+P(s)K(s)}$$: Sensitivity

$|S(j\omega)|$ small: Good Feedback Performance

Sensitivity as Feedback Performance in SISO Systems

Insensitivity to Plant Variations [SP05, p. 23]



$$\underset{(r \to y)}{G_{yr}} = \frac{PK}{1+PK} \quad \Rightarrow \quad \frac{dG_{yr}}{G_{yr}} = S\frac{dP}{P}$$

$$\left(\frac{dG_{yr}}{dP} = \frac{K}{(1+PK)^2} = \frac{SPK}{P(1+PK)} = S\frac{G_{yr}}{P} \right)$$

$$\underset{(d_i \to y)}{G_{yd_i}} = \frac{P}{1+PK} \quad \Rightarrow \quad \frac{dG_{yd_i}}{G_{yd_i}} = S\frac{dP}{P}$$

$$\left(\frac{dG_{yd_i}}{dP} = \frac{1}{(1+PK)^2} = \frac{SP}{P(1+PK)} = S\frac{G_{yd_i}}{P} \right)$$

$|S(j\omega)|$ small : Good Feedback Performance (Absolute Value)

⇒ MIMO ?

H_∞ Norm as System Gain [SP05] (p. 158)

System Gain $\|G(s)\|_\infty = \max_{\omega} \bar{\sigma}(G(j\omega))$

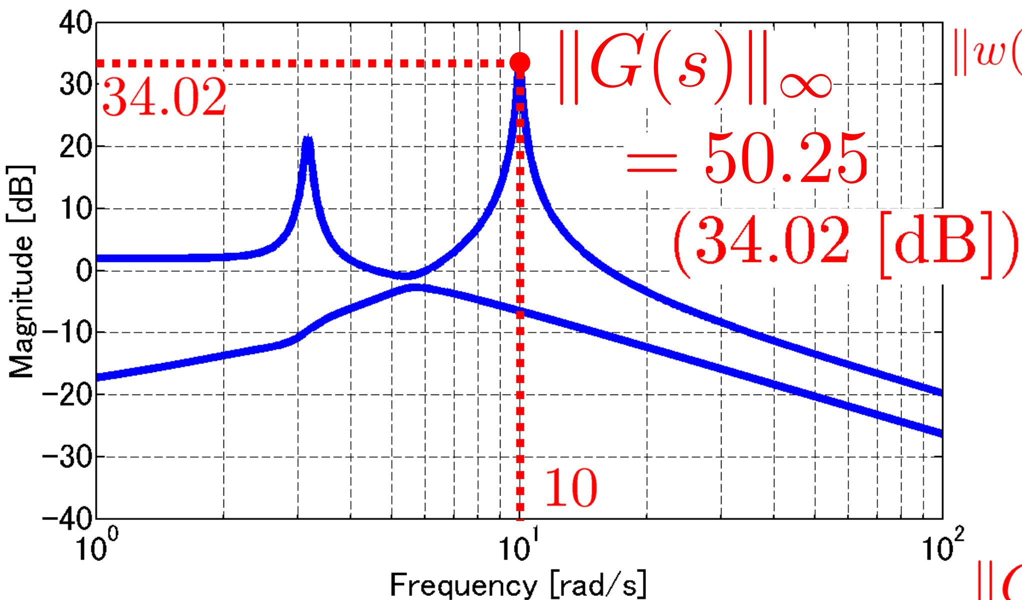
$G(s) \in \mathcal{S}$: Proper stable system

[Ex.]

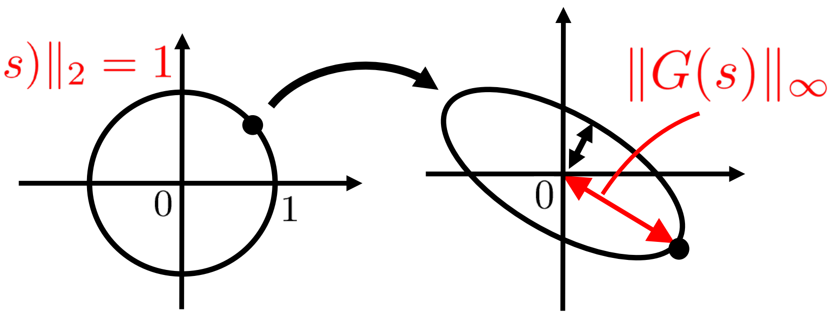
$$G(s) = \begin{bmatrix} \frac{10(s+1)}{s^2+0.2s+100} & \frac{1}{s+1} \\ \frac{s+2}{s^2+0.1s+10} & \frac{5(s+1)}{(s+2)(s+3)} \end{bmatrix}$$

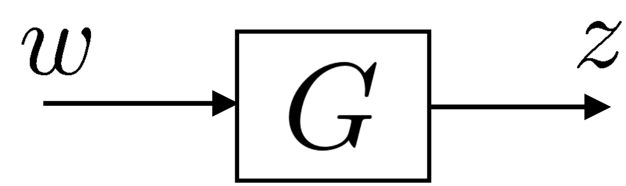
G. H. Hardy

MATLAB Command
hinfG = normhinf(G)



σ- plot





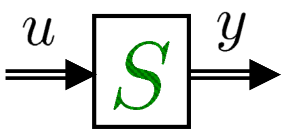
$$\|G(s)\|_\infty = \max_{\|\omega\|=1} \|z\|_2 = \max_{\omega \neq 0} \frac{\|z\|_2}{\|\omega\|_2}$$

Difference between the H_2 and H_∞ norms [SP05, pp. 75, 159]

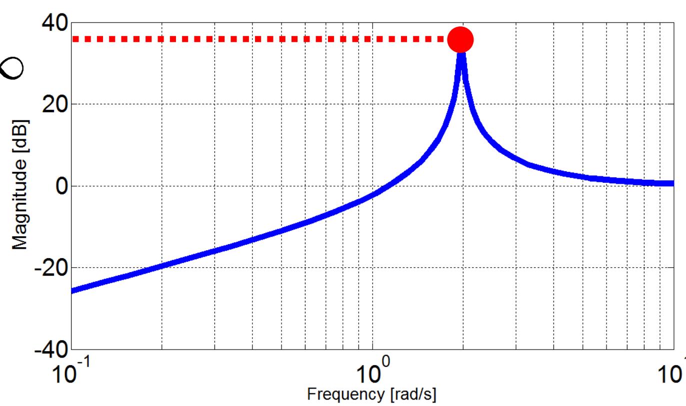
Minimizing H_∞ norm

Push down
"peak of maximum singular value"

Worst direction, worst frequency

$\|u(t)\|_2 = 1$  $\max \|y(t)\|_2$

$\|S\|_\infty$



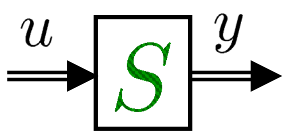
Multiplicative property

$$\|A(s)B(s)\|_\infty \leq \|A(s)\|_\infty \|B(s)\|_\infty$$

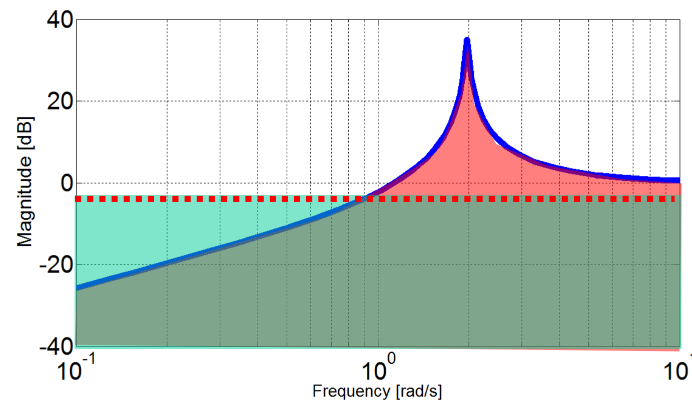
Minimizing H_2 norm (LQG)

Push down "whole thing"
(all singular values over all frequencies)

Average direction, average frequency

$u = \delta(t)$  $\|y(t)\|_2$

$\|S\|_2$



Multiplicative property

$$\|A(s)B(s)\|_2 \overset{?}{\leq} \|A(s)\|_2 \|B(s)\|_2$$

Optimization in Feedback Control

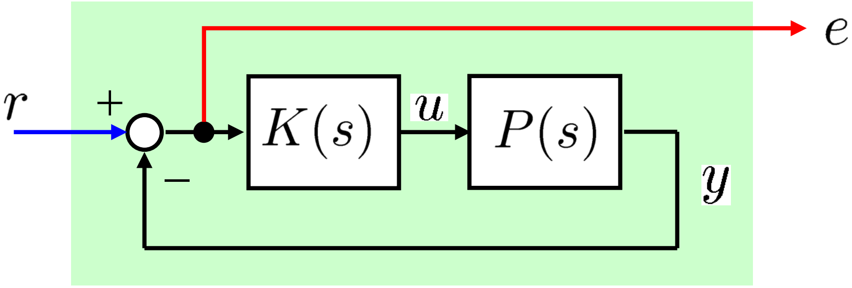
"Feedback Performance = Sensitivity"

Sensitivity optimization with H_∞ norm

$$\min_{\text{Feedback } K} \|S\| = \min_K \|(I+PK)^{-1}\|_\infty \quad \text{(System Gain)}$$

$$= \min_Q \|I - PQ\|_\infty \quad (Q\text{-parameterization})$$

Sensitivity
from Reference to Error



$$e = (I+PK)^{-1} r = Sr$$

H_∞?

A. H. Haddad (Ed.), IEEE TAC 1987

George Zames（1934-1997）

G.Zames, IEEE TAC, 26, 1981

Frustration with LQG control (H_2 control)

- Formulation of the optimization problem not on time domain but on frequency domain

H_∞ control

1939 The World War II occurred.
Escaping to Europe through Lithuania
Witnessed by Soviet’s tank
Through Russia, Siberia and Japan sea,

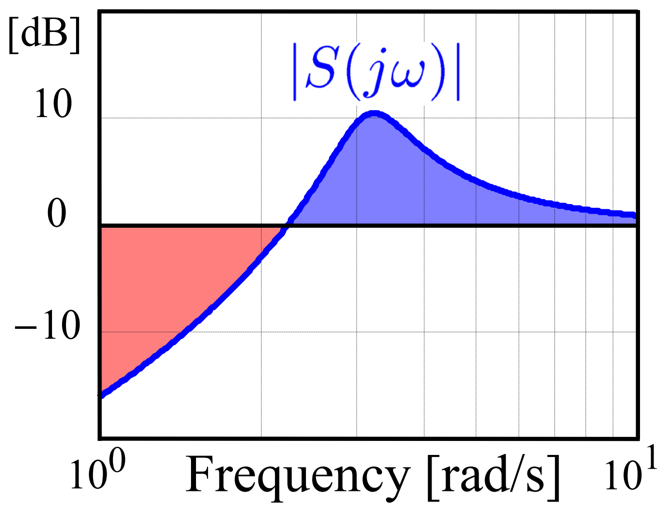
1941 Arrival in Kobe
Sugihara “Sempo” Chiune, consular officer of Japan, helped him a lot.
Leaving for Canada

Bode Sensitivity Integrals (Waterbed Effects) for Stable Plant

$$\int_0^{\infty} \log|\det S(j\omega)|d\omega = 0 \qquad |\det S| < 1 \quad |\det S| > 1$$

[SP05, p. 167, p. 223]

There exists a frequency range over which the magnitude of the sensitivity function exceeds 1 if it is to be kept below 1 at the other frequency range.



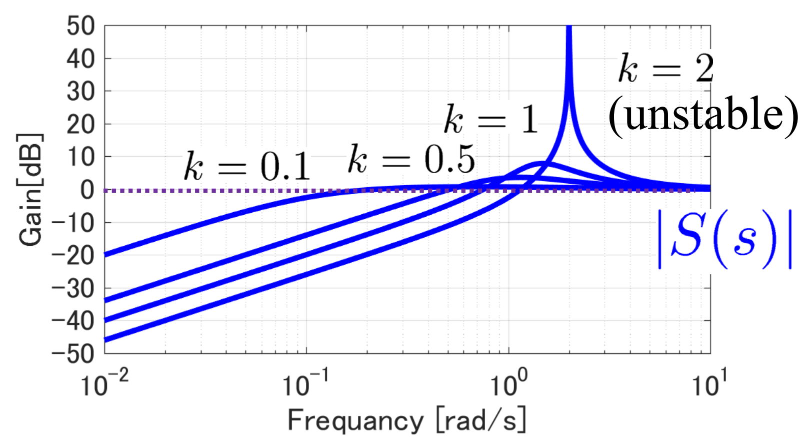
Dirt

[SP05, Ex., p. 170]

$$P(s) = \frac{2-s}{2+s},$$ RHP(Right-Half Plane) Zero

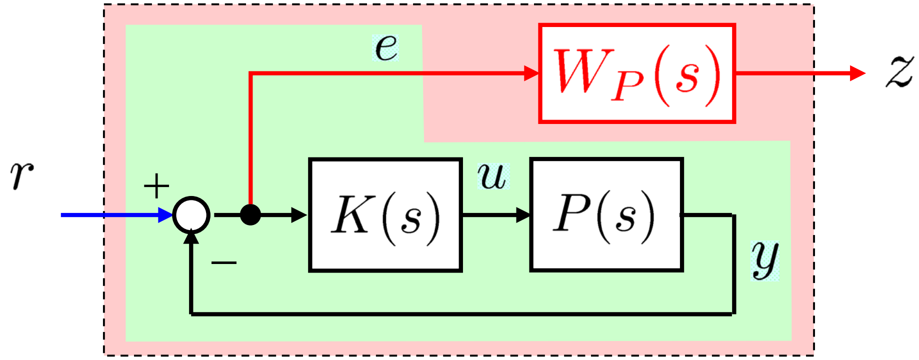
$$K(s) = \frac{k}{s},$$

$$S(s) = \frac{1}{1+P(s)K(s)}$$



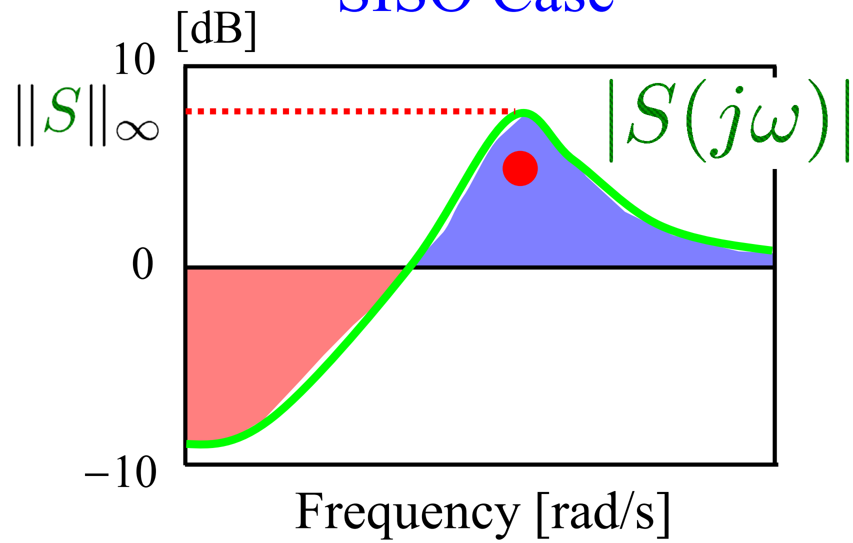
Weighted Sensitivity [SP05, p. 60]

$\|S\|_\infty$ Small?



$$e = (I + PK)^{-1} r = Sr$$

SISO Case



Waterbed Effects

$\|W_P S\|_\infty$ Small! Intractable → Tractable!

W_P : Performance weight transfer function matrix [SP05, pp. 62, 80]

$$W_P(s) = \begin{bmatrix} w_{p1}(s) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & w_{pn}(s) \end{bmatrix} \left(= \begin{bmatrix} w_p(s) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & w_p(s) \end{bmatrix} \right)$$

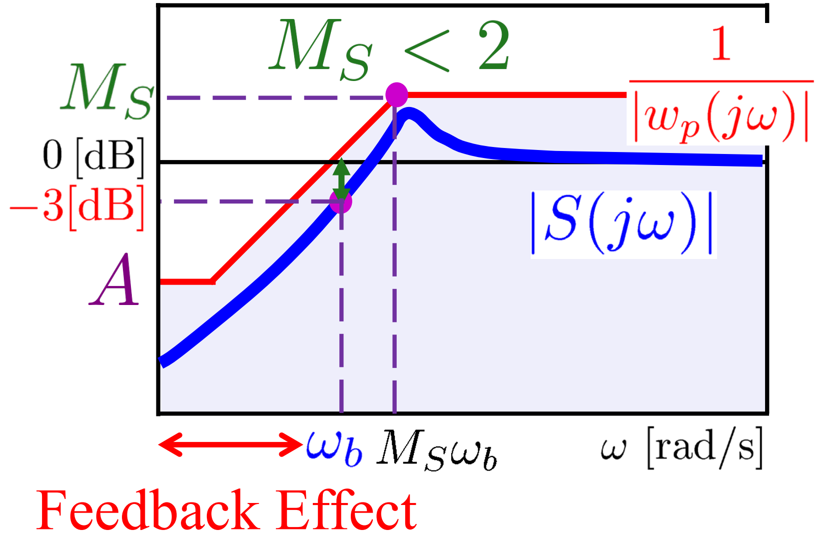
How to specify $w_p(s)$?

Performance Weight W_P [SP05, pp. 62, 80]

First-order Performance Weight

$$w_p(s) = \frac{\frac{1}{M_S}s + \omega_b}{s + \omega_b A}$$

$$w_p(s) = \frac{\left(s/Ms^{1/n} + \omega_b\right)^n}{\left(s + \omega_b A^{1/n}\right)^n}$$



ω_b : the frequency at which the asymptote of $1/|w_p(j\omega)|$ crosses 1, and the bandwidth requirement approximately

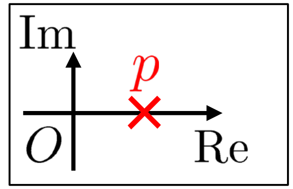
M_S: $1/|w_p(j\omega)|$ at high frequencies ($M_S < 2$: Rule of thumb)

A : $1/|w_p(j\omega)|$ at low frequencies

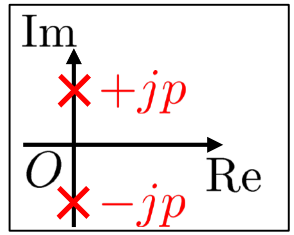
Stabilization and Performance

Unstable Plant [SP05 Sec 5.9]

Real RHP Poles: $2p < \omega_c$

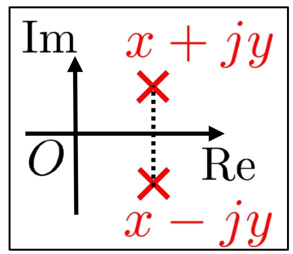


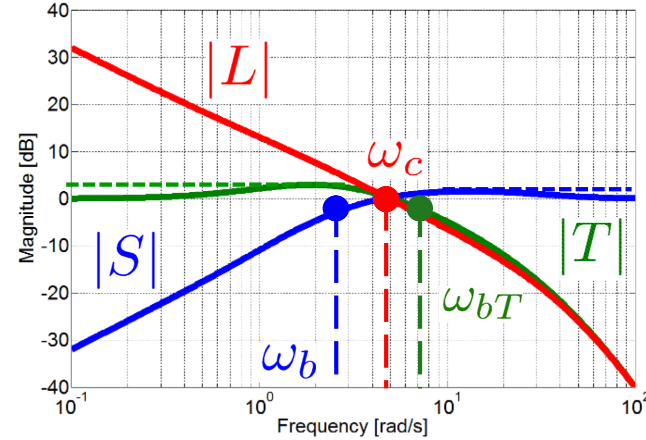
Imaginary Poles: $1.15|p| < \omega_c$

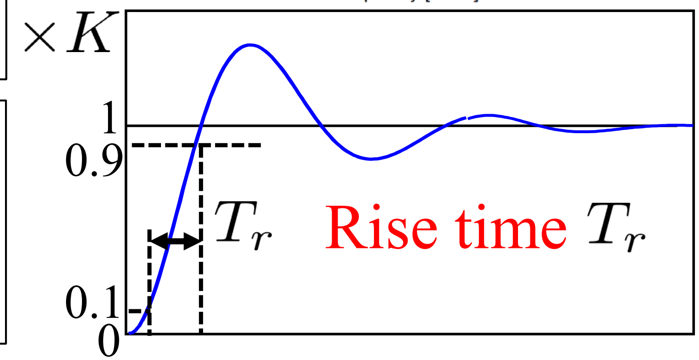


Complex RHP Poles:

$$0.67\left(x + \sqrt{4x^2 + 3y^2}\right) < \omega_c$$







Stable Plant

First-order System

$$G_1(s) = \frac{K}{Ts+1} \quad K > 0 \quad T > 0$$

Rise time $T_r = (\ln 9)T \approx 2.2T$

$$\frac{2.2}{T_r} \leq \frac{1}{T} \leq \omega_c$$

Second-order System

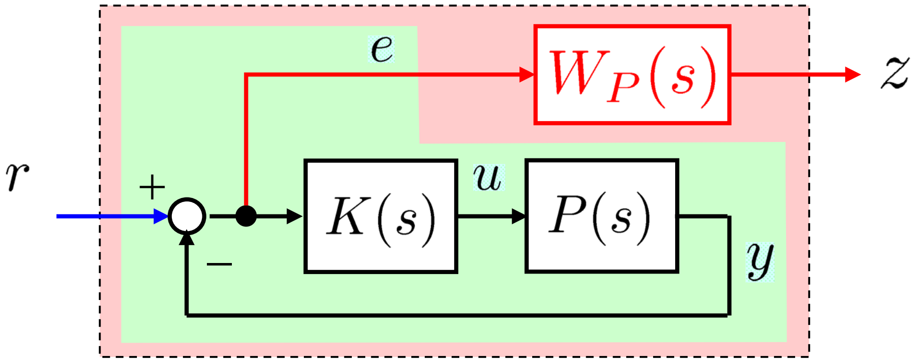
$$G_2(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \omega_n > 0 \quad \zeta \geq 0$$

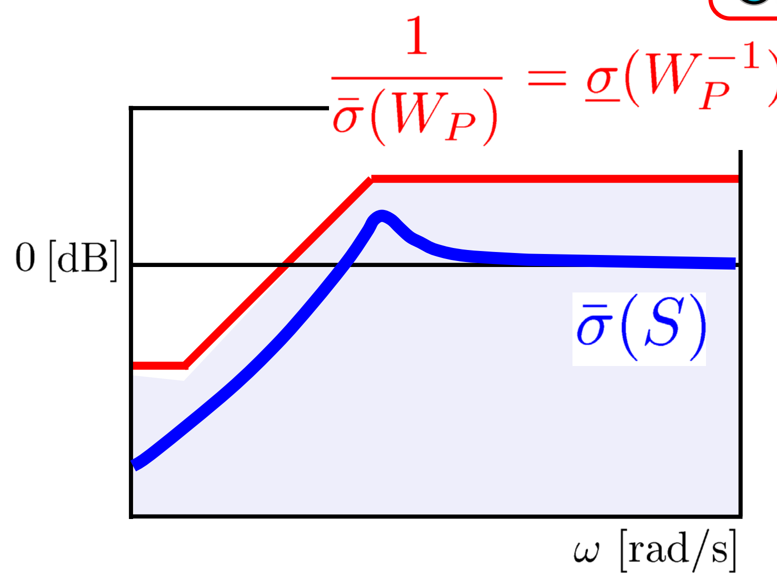
Rise time $T_r = \frac{\pi/2 + \arcsin\zeta}{\omega_n\sqrt{1-\zeta^2}}$

$$\zeta = 0.2 \quad \frac{1.8}{T_r} \leq \omega_r \leq \omega_c$$

[QZ07] L. Qiu and K. Zhou (2007) *Introduction to Feedback Control*, Prentice Hall.

Nominal Performance (NP) [SP05, p. 81]





$$\bar{\sigma}(S(j\omega)) < \frac{1}{\bar{\sigma}(W_P(j\omega))} \quad \forall\omega$$

$$\bar{\sigma}(W_P(j\omega)S(j\omega)) < 1 \quad \forall\omega$$

$$\left(\begin{array}{c} \underline{\sigma}(A)\bar{\sigma}(B) \le \bar{\sigma}(AB) \\ \bar{\sigma}(AB) \le \bar{\sigma}(A)\bar{\sigma}(B) \\ \bar{\sigma}(A^{-1}) = \dfrac{1}{\underline{\sigma}(A)} \\ \|A\|_\infty = \bar{\sigma}(A) \end{array} \right)$$

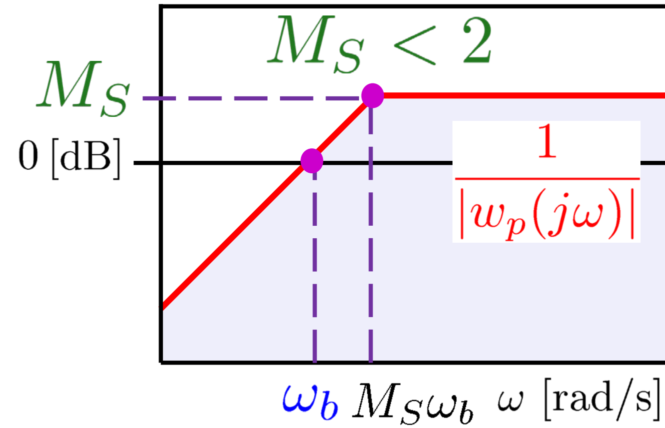
Nominal Performance (NP) Test

Given a controller K,

$$\|W_P(s)S(s)\|_\infty < 1$$

Nominal Performance Test in SISO Systems [SP05, p. 60]

$$\text{(NP)} \quad |S(j\omega)| < \frac{1}{|w_p(j\omega)|} \quad \forall \omega$$



[Ex.]

$$S = \frac{s^2 + s}{s^2 + 0.7s + 0.07}$$

$$w_p(s) = \frac{\frac{1}{M_S}s + \omega_b}{s}$$

- 1) w_{p1} $\omega_b = 0.01$, $M_S = 2$

  (NP) 👍

  ω_b : fast M_S: small

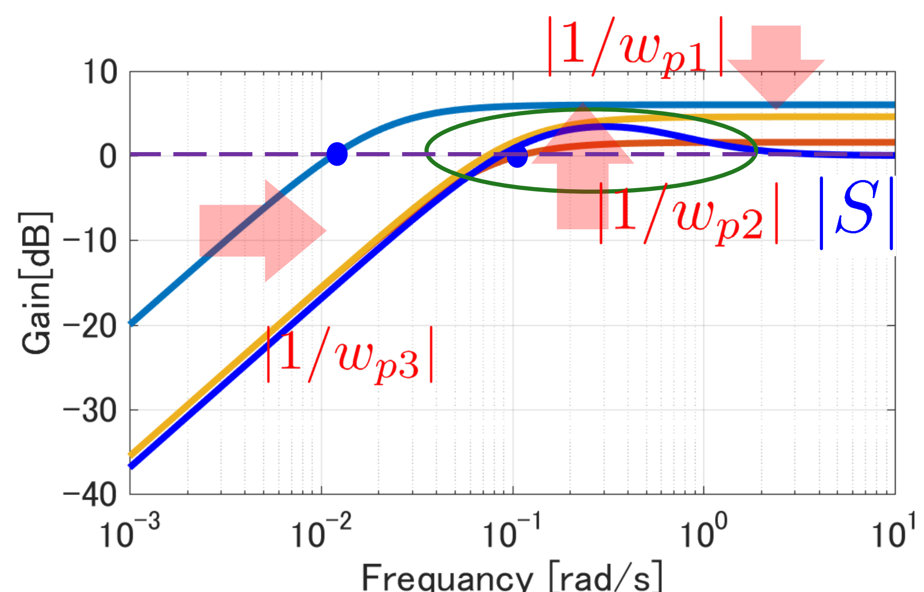
- 2) w_{p2} $\omega_b = 0.06$, $M_S = 1.2$

  (NP) 👎

  M_S: large

- 3) w_{p3} $\omega_b = 0.06$, $M_S = 1.7$

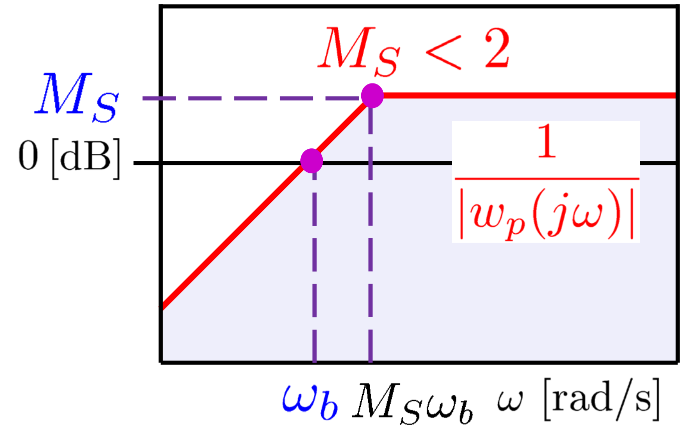
  (NP) 👍



Nominal Performance Test in SISO Systems [SP05, p. 60]

(NP) $\|W_P(s)S(s)\|_\infty < 1$

H_∞ Norm Condition



[Ex.]

$$S = \frac{s^2 + s}{s^2 + 0.7s + 0.07}$$

$$w_p(s) = \frac{\frac{1}{M_S}s + \omega_b}{s}$$

- 1) w_{p1} $\omega_b = 0.01$, $M_S = 2$

  (NP) 👍

  ω_b : fast M_S : small

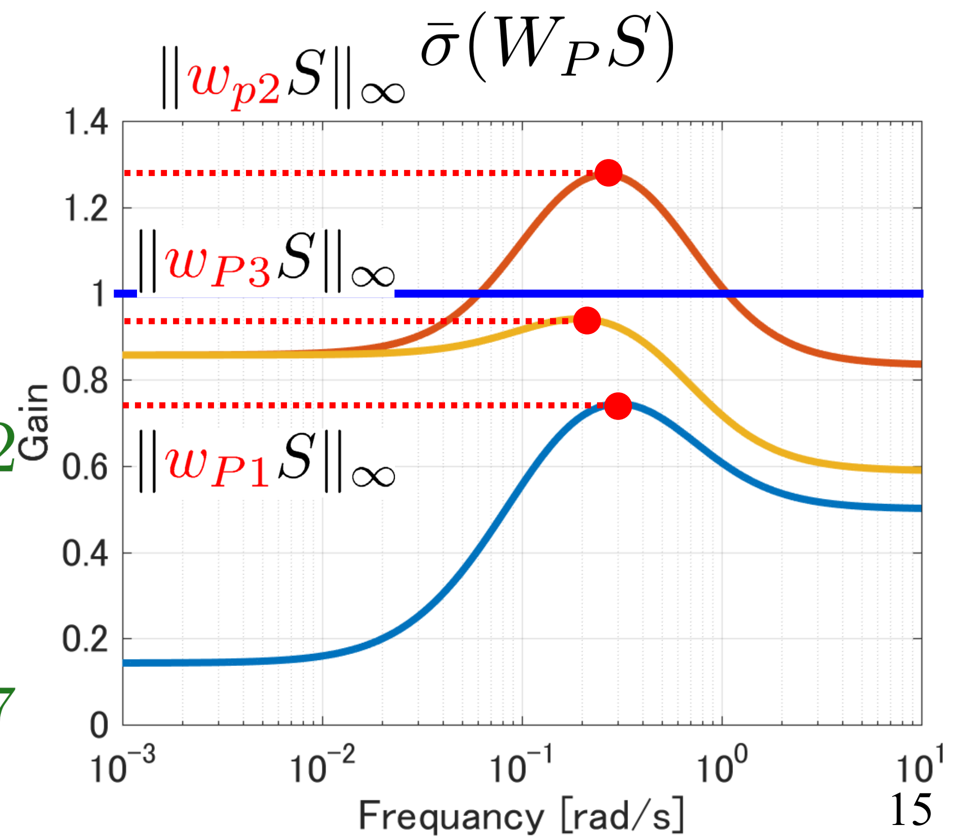
- 2) w_{p2} $\omega_b = 0.06$, $M_S = 1.2$

  (NP) 👎

  M_S : large

- 3) w_{p3} $\omega_b = 0.06$, $M_S = 1.7$

  (NP) 👍



[Ex.] Spinning Satellite: Performance Weight

Performance Weight W_P

$$W_P(s) = \begin{bmatrix} w_p(s) & 0 \\ 0 & w_p(s) \end{bmatrix} \left(= \begin{bmatrix} w_{p1} & 0 \\ 0 & w_{p2} \end{bmatrix} \right)$$

MATLAB Command

```
Ms = 2; A = 1e-2; wb = 11.5;
wP = tf([1/Ms wb], [1 wb*A]);
WP = eye(2)*wP;
figure
sigma(WP)
hold on; grid on;
```

Specifications $w_p(s) = \dfrac{\frac{1}{M_s}s + \omega_b}{s + \omega_b A}$

- Poles on the imaginary axis

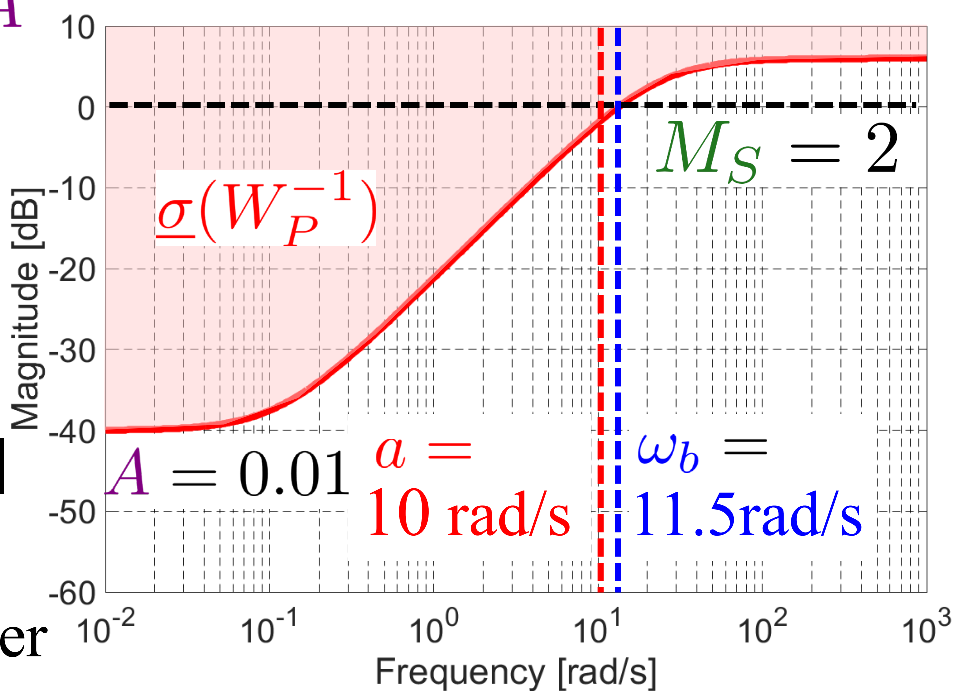
$$p = \pm aj = \pm 10\, j$$

Gain crossover frequency

$$\omega_c > 1.15|p| = 11.5 \text{ rad/s} = \omega_b$$

Phase stabilization [SP05, p. 194]

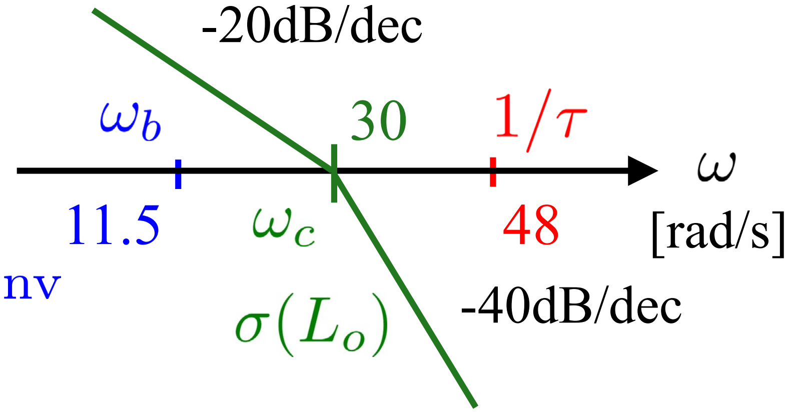
$\omega_c <$ System bandwidth of Actuator/Sensor/Controller



- $M_S \leq 2 \rightarrow M_S = 2$
- the steady state error $e \leq 0.01 \rightarrow A = 0.01$

[Ex.] Spinning Satellite: Nominal Performance

Plant $P(s)=\begin{bmatrix} \frac{s-100}{s^2+100} & \frac{10s+10}{s^2+100} \\ \frac{-10s-10}{s^2+100} & \frac{s-100}{s^2+100} \end{bmatrix}$



Controller: Inverse-based Controller $K_{\rm inv}$

$$K_{\rm inv}(s)=P^{-1}(s)\begin{bmatrix} \frac{900}{s(s+30)} & 0 \\ 0 & \frac{900}{s(s+30)} \end{bmatrix}$$

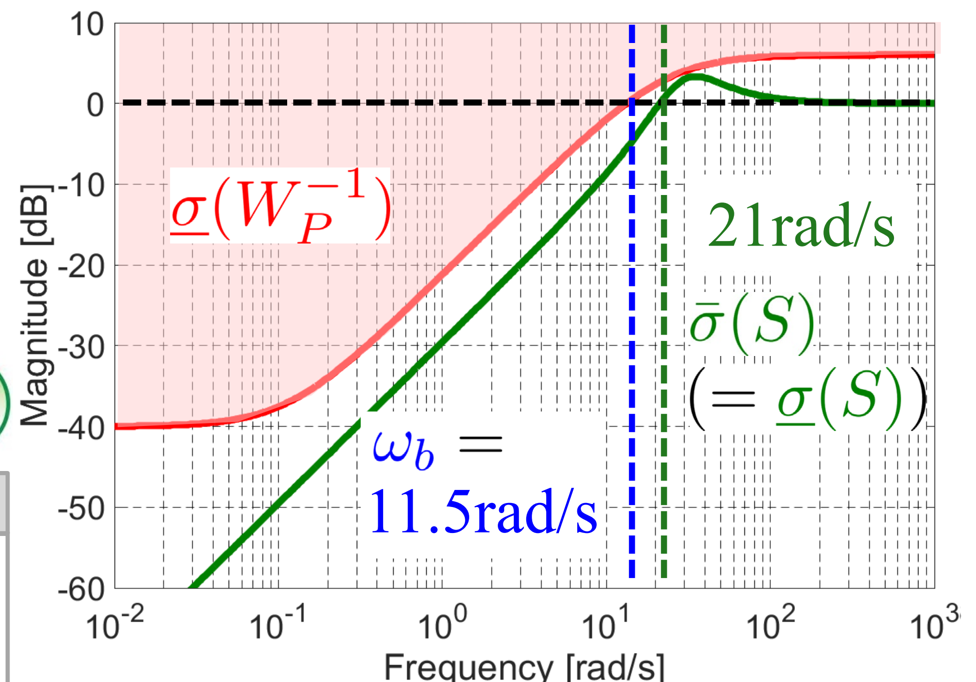
Target Loop Transfer Function

$$L(s)=PK_{\rm inv}=\frac{900}{s(s+30)}I_2$$

(Output) Sensitivity Function

$$S(s)=(I+PK_{\rm inv})^{-1}$$

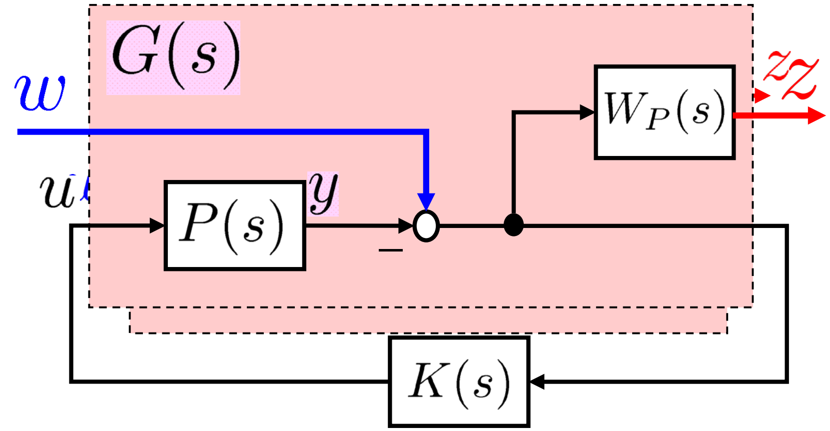
$$\|W_PS\|_\infty = 0.8935 < 1 \quad \text{NP}$$



MATLAB Command

```
KI = inv(Pnom)*tf([1],[1 30 0])*diag([900 900]);
FI = loopsens(Pnom,KI);
sigma(FI.So) ;
hinfSo = normhinf(WP*FI.So)
```

Sensitivity Minimization



Optimal Sensitivity Problem

Find a stabilizing controller K which minimize / make smaller $\|W_P(s)S(s)\|_\infty$

Intractable

Linear Fractional Transformation (LFT)

Sensitivity Minimization Problem

H_∞ Control

Given $\gamma > \gamma_{min}$, find all stabilizing controllers K such that

$\|W_P(s)S(s)\|_\infty < \gamma$ γ -iteration

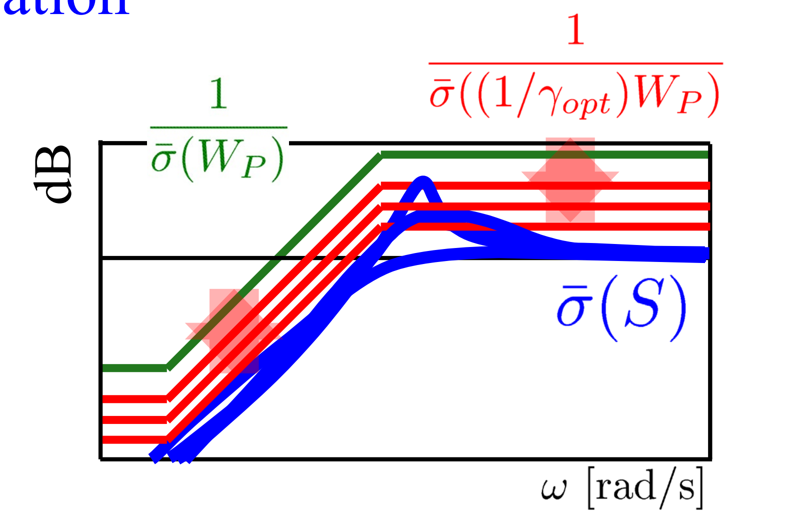
1) $\|W_P S\|_\infty < \gamma_1$  $^\exists K_1$

2) $\|W_P S\|_\infty < \gamma_2$  no K_2

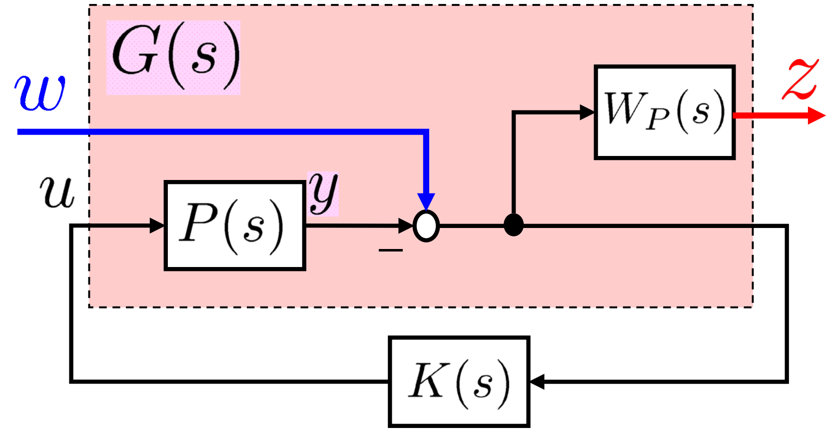
3) $\|W_P S\|_\infty < \gamma_3$  $^\exists K_3$

$\vdots$

$\frac{1}{\gamma_{opt}} W_P$ $\left[\because Q \text{ Parameterization}\right]$



Sensitivity Minimization



Optimal Sensitivity Problem

Find a stabilizing controller K which make smaller $\|W_P(s)S(s)\|_\infty$

tractable

Linear Fractional Transformation (LFT)

Sensitivity Minimization Problem

H_∞ Control

Given $\gamma > \gamma_{min}$, find all stabilizing controllers K such that

$$\|W_P(s)S(s)\|_\infty < \gamma$$

γ -iteration

1) $\|W_P S\|_\infty < \gamma_1$  $^\exists K_1$

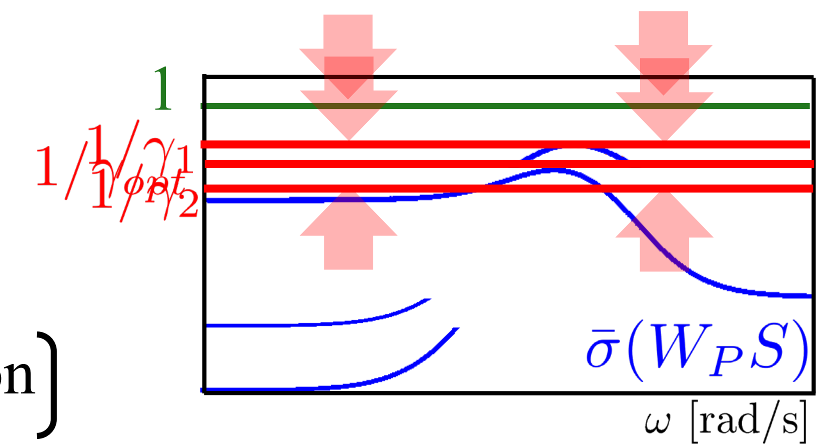
2) $\|W_P S\|_\infty < \gamma_2$  no K_2

3) $\|W_P S\|_\infty < \gamma_3$  $^\exists K_3$

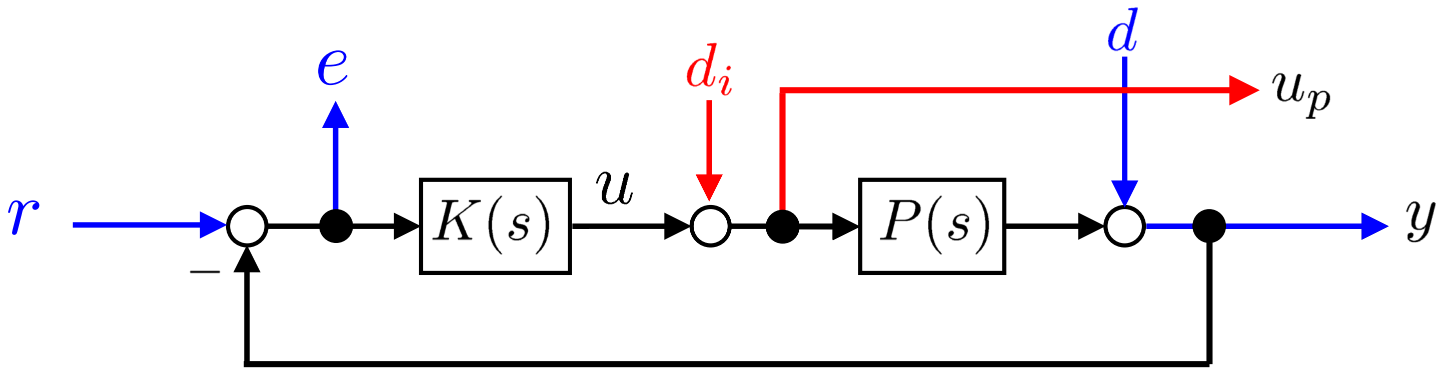
$\vdots$

$\frac{1}{\gamma_{opt}} W_P$

$\left[\because Q \text{ Parameterization}\right]$



Sensitivity for MIMO Systems [SP05, p. 70]



Sensitivity to Output Disturbance d

Output Sensitivity Function: $S_o(s) = (I + P(s)K(s))^{-1}$

Sensitivity to Input Disturbance d_i

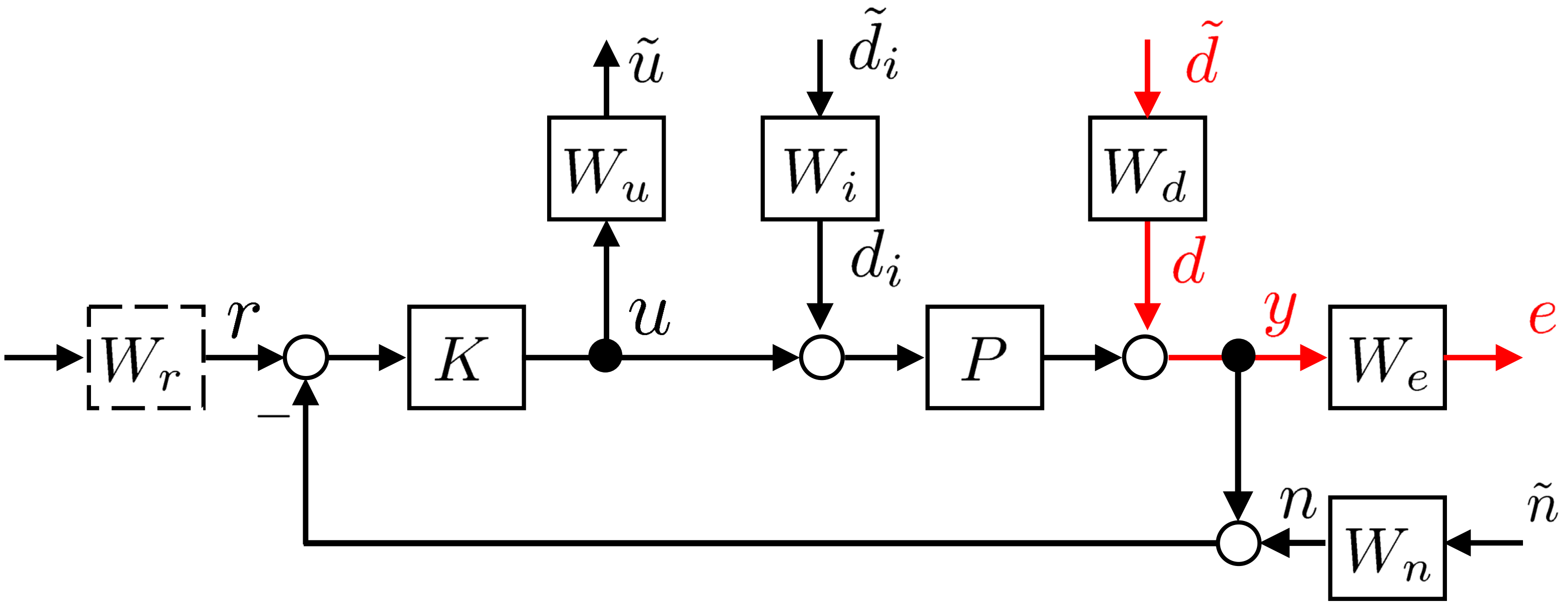
Input Sensitivity Function: $S_i(s) = (I + K(s)P(s))^{-1}$

For SISO Systems $S_i = S_o$

but for MIMO Systems $PK \neq KP \rightarrow S_i \neq S_o$

Good disturbance rejection at output does not always mean good rejection at input

Standard Feedback Configuration with Weights [SP05, p. 363]



Sensitivity Minimization Problem

$$\text{find } K(s) \text{ s.t. } \|W_e(s)S_o(s)W_d(s)\|_\infty < \gamma$$

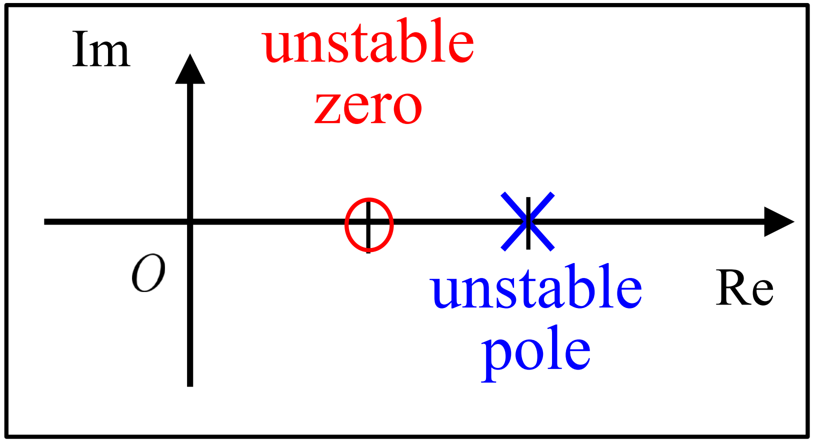
Remarks on Fundamental Limitations

Gunter Stein

"Respect the unstable"

Bode lecture, CDC, 1989

Control Systems Magazine, 23(4):12-25, 2003.

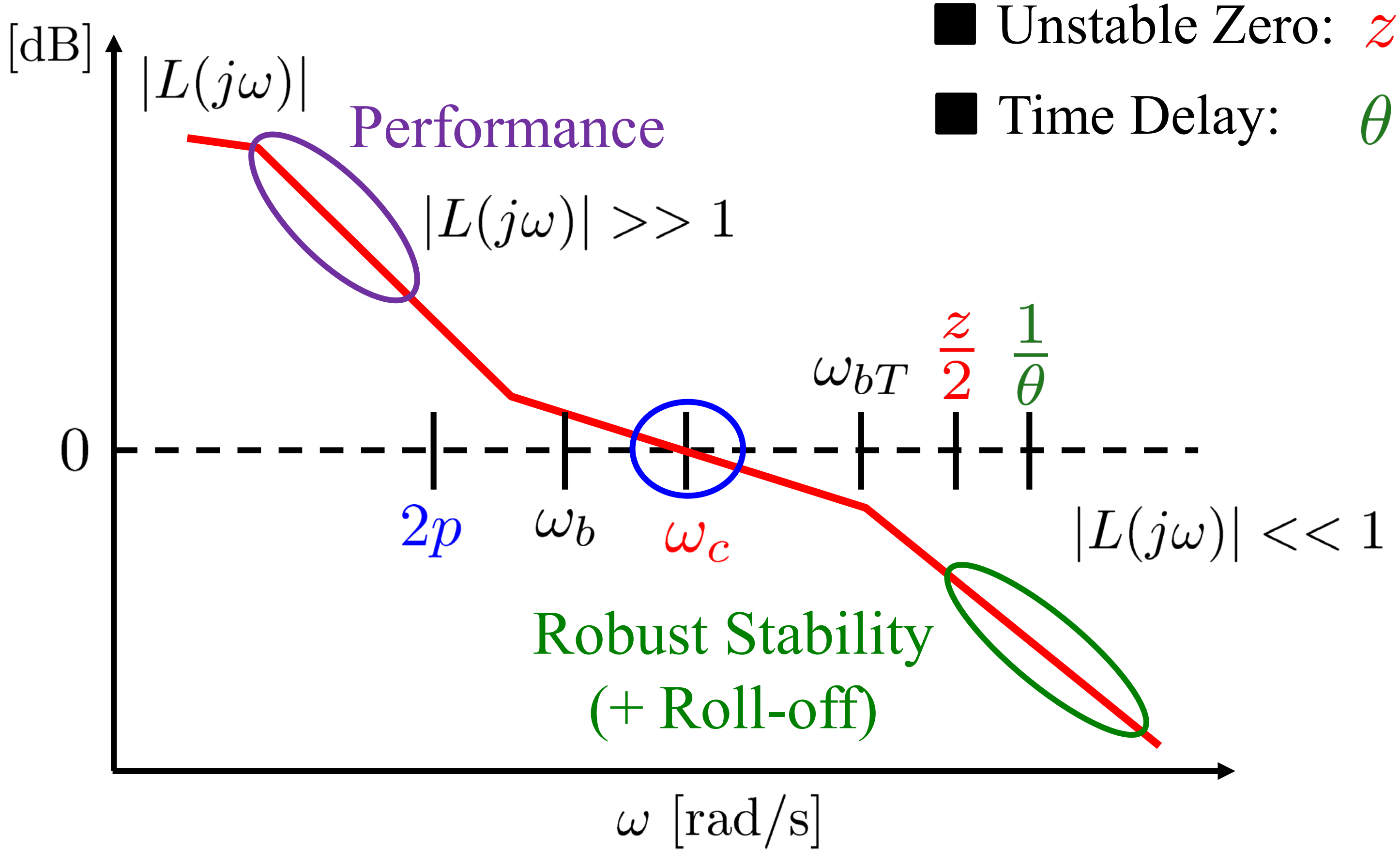


Unstable Zero

- Time Delay
- Wrong Sensor Placement

SISO Loop Shaping [SP05, pp. 41, 42, 343]

- Unstable Pole: p
- Unstable Zero: z
- Time Delay: θ



Loop Shaping

gives us graphical interpretation

- Bode Plot
- System Gain

RHP Poles/Zeros, Time Delays and Sensitivity in SISO Systems

6 ~ 7

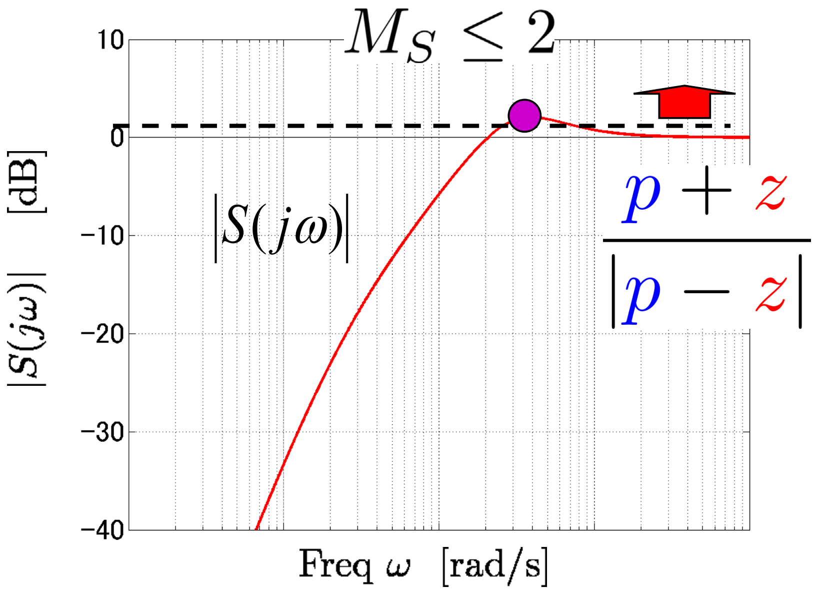
For systems with a RHP pole p and RHP zero z (or a time delay θ), any stabilizing controller gives sensitivity functions with the property

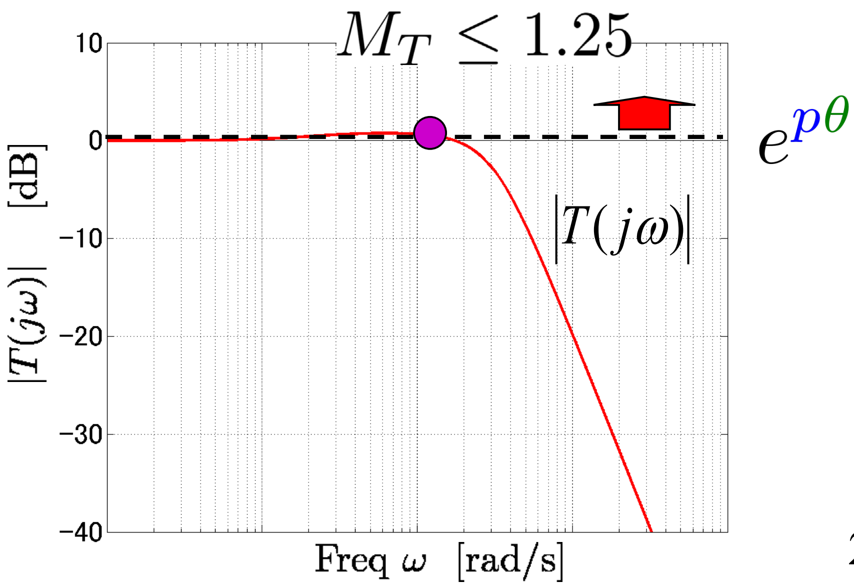
$$M_S = \sup_\omega |S(j\omega)| \geq \frac{p+z}{|p-z|}$$

The zero and the pole must be sufficiently far apart

$$M_T = \sup_\omega |T(j\omega)| \geq e^{p\theta}$$

The product of RHP pole and time delay must be sufficiently small





2. Nominal Performance

2.1 Weighted Sensitivity

[SP05, Sec. 2.8, 3.3, 4.10, 6.2, 6.3]

2.2 Nominal Performance [SP05, Sec. 2.8, 3.2, 3.3]

2.3 Sensitivity Minimization [SP05, Sec. 3.2, 3.3, 9.3]

2.4 Remarks on Fundamental Limitations

[SP05, Sec. 6.2]

Reference:

[SP05] S. Skogestad and I. Postlethwaite, ***Multivariable Feedback Control; Analysis and Design,*** Second Edition, Wiley, 2005.

3. Robustness and Uncertainty

3.1 Why Robustness? [SP05, Sec. 4.1.1, 7.1, 9.2]

3.2 Representing Uncertainty [SP05, Sec. 7.2, 7.3, 7.4]

3.3 Uncertain Systems [SP05, Sec. 8.1, 8.2, 8.3]

3.4 Systems with Structured Uncertainty [SP05, Sec. 8.2]

Reference:

[SP05] S. Skogestad and I. Postlethwaite, ***Multivariable Feedback Control; Analysis and Design,*** Second Edition, Wiley, 2005.

Norm [SP05, A.5]

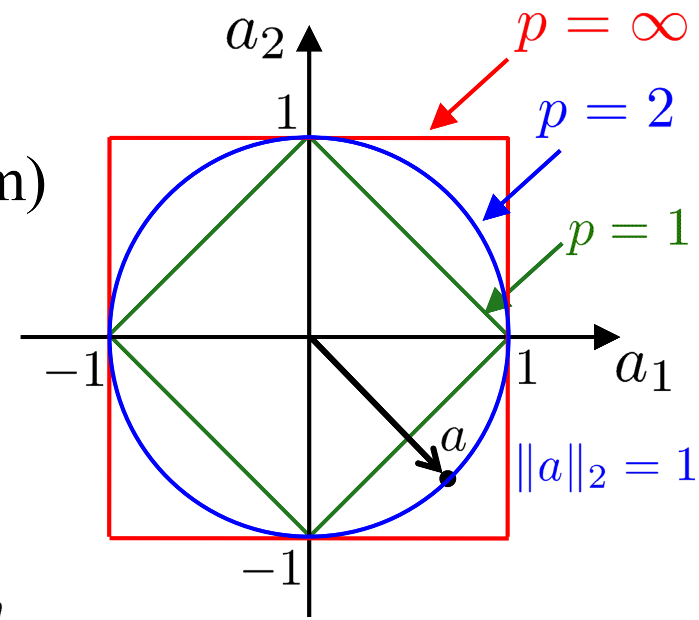
Key properties

1. Non-negative $\|e\| \geq 0$
2. Positive $\|e\| = 0$ iff $e = 0$
3. Homogeneous $\|\alpha e\| = |\alpha|\|e\|$, $\forall \alpha$:scalar
4. Triangle inequality $\|e_1 + e_2\| \geq \|e_1\| + \|e_2\|$

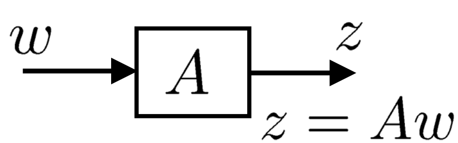
Vector Norm [Ex.]

$$\|a\|_2 = \sqrt{\sum_i |a_i|^2} \text{ (Euclidean Vector Norm)}$$

$$\|a\|_1 = \sum_i |a_i|, \quad \|a\|_\infty = \max_i |a_i|$$

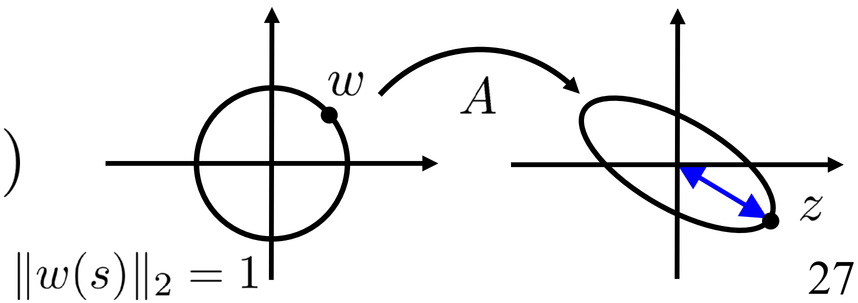


(Induced) Matrix Norm



[Ex.]

$$\|A\|_{i2} = \max_{\omega \neq 0} \frac{\|z\|_2}{\|\omega\|_2} = \bar{\sigma}(A)$$

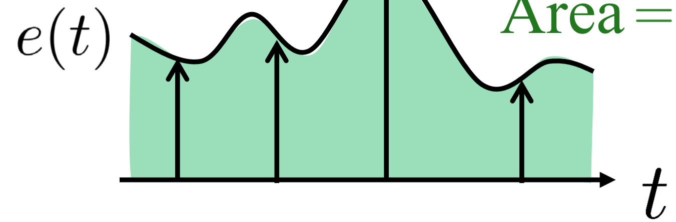


Norm [SP05, A.5]

$\|e(t)\|_\infty$

Energy $= \|e(t)\|_2$

Area $= \|e(t)\|_1$



Signal Norm

[Ex.]

"Energy of signal"
($\mathcal{L}_2$-norm, $\mathcal{L}$: Lebesgue space)

$$\|e(t)\|_2^2 = \int_{-\infty}^{\infty} \sum_i |e_i(t)|^2 dt$$

Integral absolute error

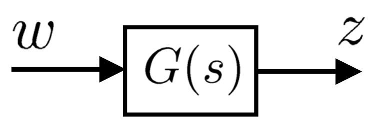
$$\|e(t)\|_1 = \int_{-\infty}^{\infty} \sum_i |e_i(t)| dt$$

"maximum value over time"

$$\|e(t)\|_\infty = \max_t \left(\max_i |e_i(t)| \right)$$

System Norm

MIMO



$$\|G(s)\|_\infty = \max_{\omega \neq 0} \frac{\|z\|_2}{\|\omega\|_2} = \max_\omega \bar{\sigma}(G(j\omega))$$ (System Gain)

$$\|G(s)\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{tr}(G(j\omega)^H G(j\omega)) d\omega} = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_i \sigma_i^2(G(j\omega)) d\omega}$$

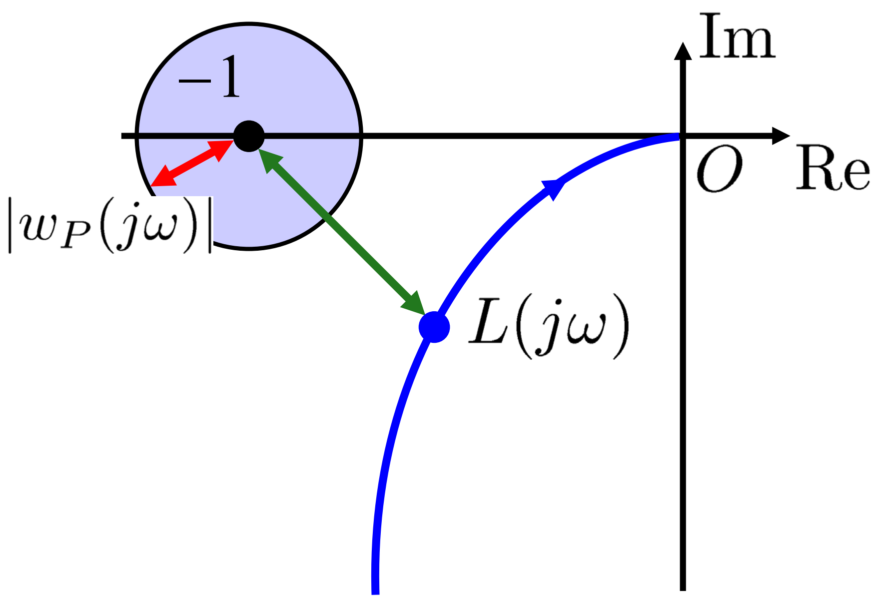
Nominal Performance in SISO Systems

$$|w_P S| < 1 \ {}^{\forall}\omega \quad \Leftrightarrow \quad |w_P| < |1+L| \ {}^{\forall}\omega$$

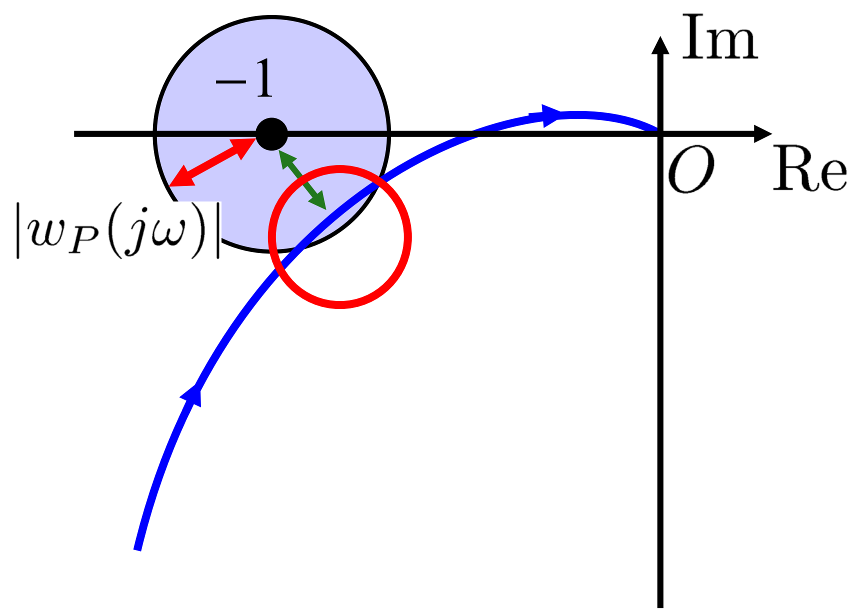
$$\left(S = \frac{1}{1+PK} = \frac{1}{1+L} \right)$$

Nyquist Plot [SP05, p. 281]

$|w_P| < |1+L| \ {}^{\forall}\omega$



$|w_P| > |1+L| \ {}^{\exists}\omega$



L should be away from $(-1, 0)$ by $|w_P|$

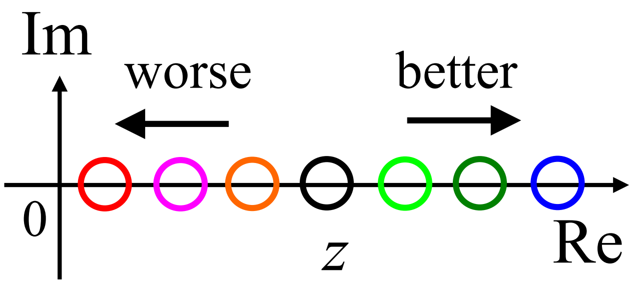
Fundamental Limitations [SP05, pp. 183]

Bound on the Crossover Frequency ω_c

RHP (Right half-plane) Zero z $\quad \omega_c < \frac{z}{2}$

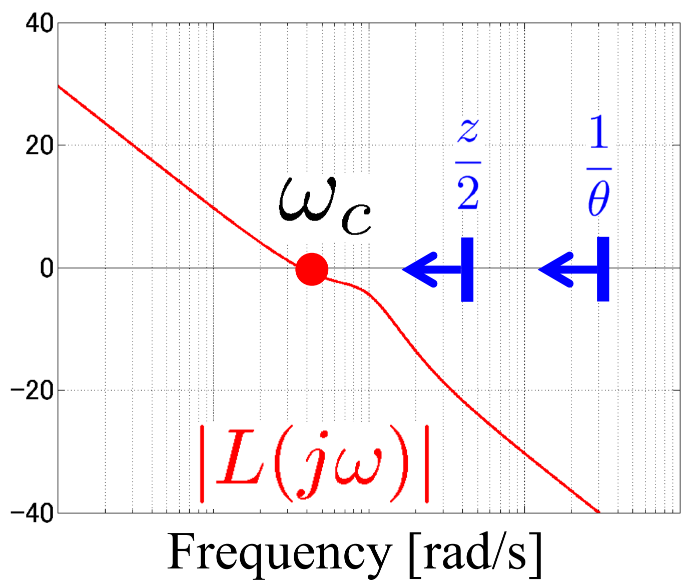
Fast RHP Zeros (z large): Loose Restrictions

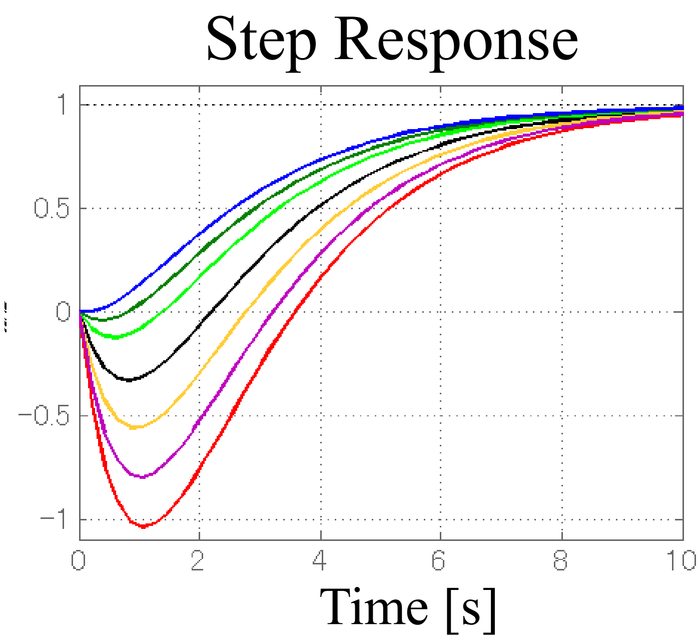
Slow RHP Zeros (z small): Tight Restrictions



Unstable zero

Time Delay θ $\quad \omega_c < \frac{1}{\theta}$





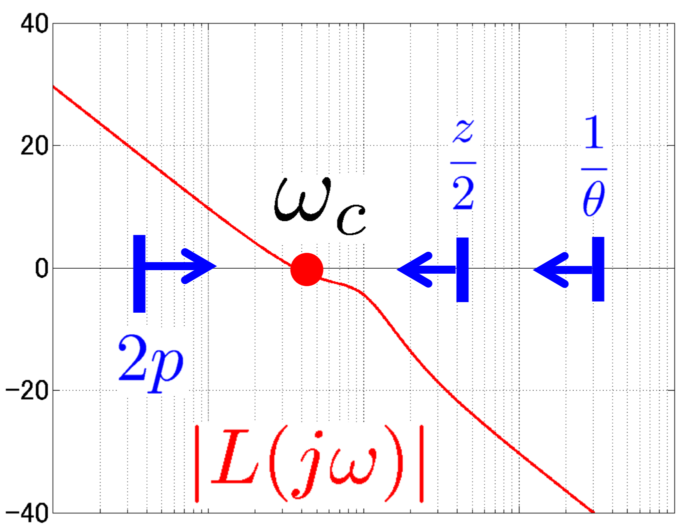
Fundamental Limitations [SP05, pp. 192, 194]

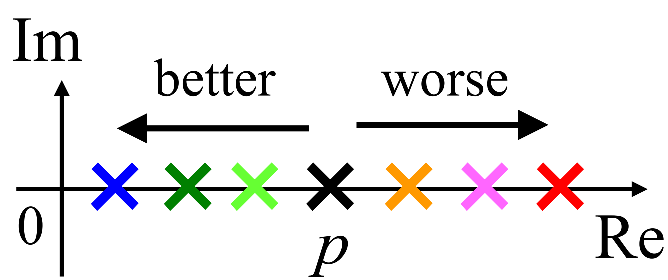
Bound on the Crossover Frequency ω_c

RHP (Right half-plane) Pole p $\omega_c > 2p$

Slow RHP Poles (p small): Loose Restrictions

Fast RHP Poles (p large): Tight Restrictions



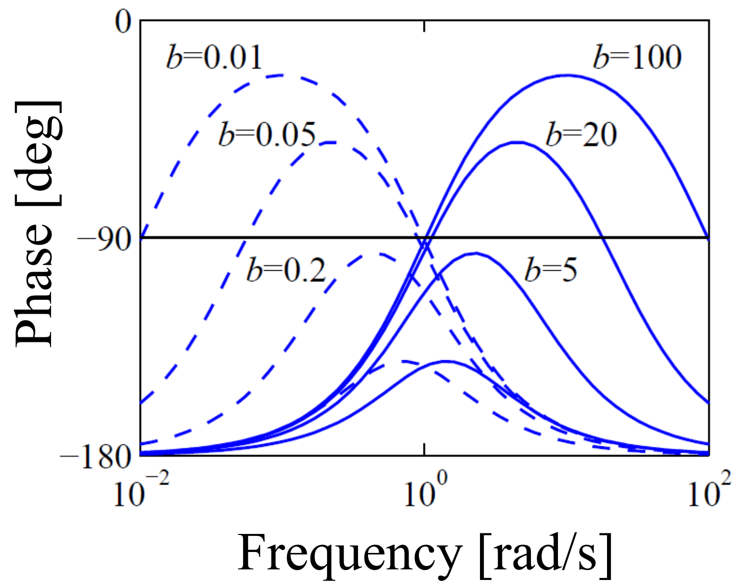


Unstable pole

Poles on imaginary axis $\pm pj$ $\omega_c > 1.15|p|$

All-pass system ($p = 1$, $z = b$, θ)

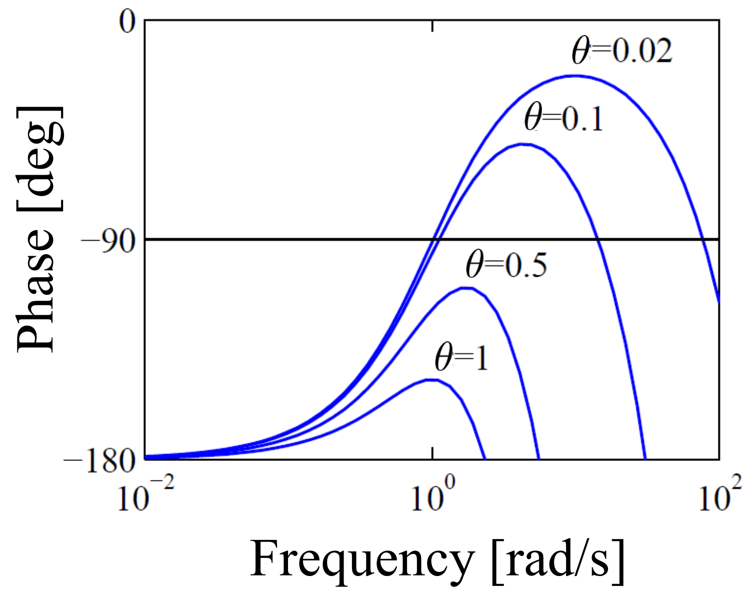
$$P(s) = \frac{b - s}{s - 1}$$



$$\frac{z}{p} < \frac{1}{6} \quad \text{or} \quad 6 < \frac{z}{p}$$

The zero and the pole must be sufficiently far apart

$$P(s) = \frac{e^{-s\theta}}{s - 1}$$



$$p\theta < 0.3$$

The product of RHP pole and time delay must be sufficiently small

Fundamental Limitations: Sensitivity in MIMO Systems

[SP05, Sec. 6.2]

Algebraic Constraint $S + T = I$

$$\Rightarrow \quad \begin{aligned} |\bar{\sigma}(S) - 1| &\le \bar{\sigma}(T) \le \bar{\sigma}(S) + 1 \\ |\bar{\sigma}(T) - 1| &\le \bar{\sigma}(S) \le \bar{\sigma}(T) + 1 \end{aligned}$$

$$\Rightarrow \quad |\bar{\sigma}(S) - \bar{\sigma}(T)| \le 1$$

$\bar{\sigma}(S)$ is large if and only if $\bar{\sigma}(T)$ is large

Fundamental Limitations: Bounds on Peaks in MIMO Systems

[SP05, Sec. 6.3]

$$M_{S,\min} \triangleq \min_K \|S\|_\infty \ , \ M_{T,\min} \triangleq \min_K \|T\|_\infty$$

One RHP Pole and One RHP Zero

$$M_{S,\min} = M_{T,\min} = \sqrt{\sin^2\phi + \frac{|z+p|^2}{|z-p|^2}\cos^2\phi}$$

$\phi = \cos^{-1}|y_z^H y_p|$ $\quad y_z, y_p$:Pole and Zero Direction [SP05, 4.4, 4.5]