Robust Control

Spring, 2019

Instructor: Prof. Masayuki Fujita (S5-303B)

4th class

Tue., 7th May, 2019, 10:45～12:15,

S423 Lecture Room

4. Robust Stability and Loop Shaping

4.1 Robust Stability and Robust Stabilization

[SP05, Sec. 7.5, 8.4, 8.5]

4.2 Mixed Sensitivity and Loop Shaping

[SP05, Sec. 2.6, 2.8, 9.1]

4.3 1st Report

Reference:

[SP05] S. Skogestad and I. Postlethwaite,
Multivariable Feedback Control; Analysis and Design,
Second Edition, Wiley, 2005.

Robust Stability

Uncertain Plant

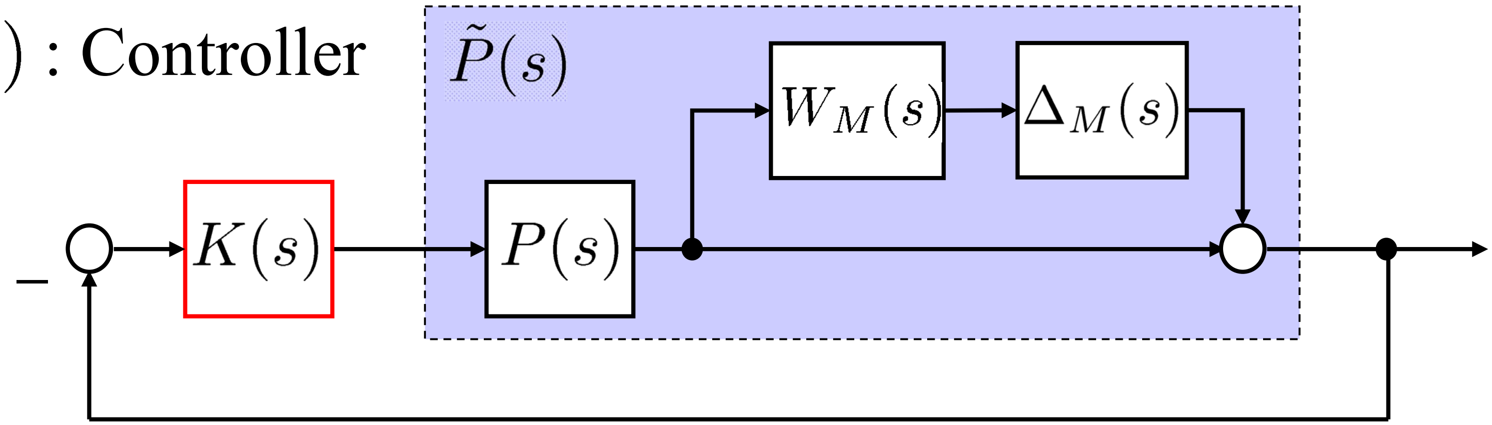
$$\tilde{P}(s) = (I + \Delta_M(s) W_M(s)) P(s) \qquad \|\Delta_M\|_\infty \le 1$$

$\tilde{P}(s) \in \Pi_0$ $\quad \Pi_0$: A set of plant models

$P(s)$: Nominal plant model

$W_M(s)$: Uncertainty Weight

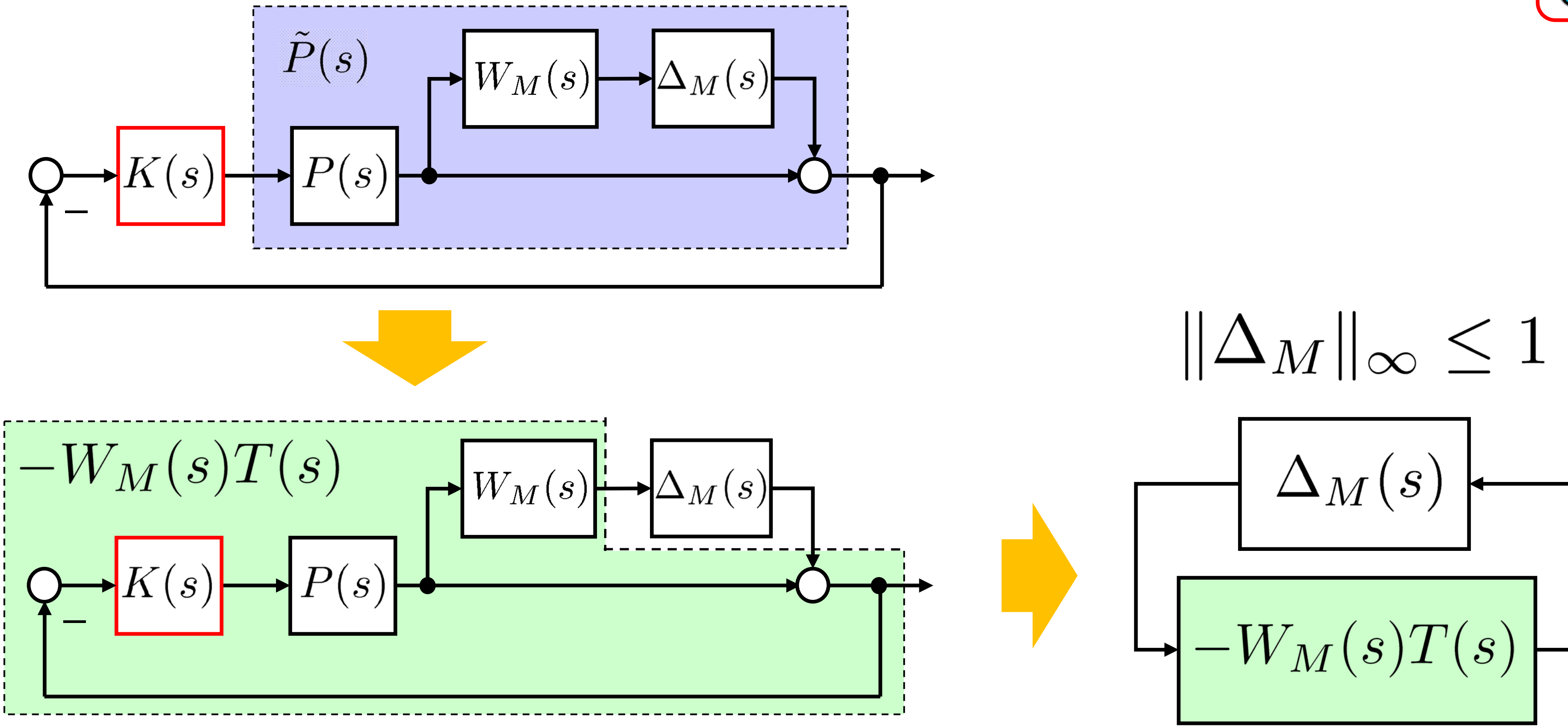
$K(s)$: Controller



Robust Stability (RS) Test [SP05, p. 300]

Given a controller K, determine whether the system remains stable for all plants in the uncertain set.

Robust Stability (RS) [SP05, pp. 276, 299]



Robust Stability (RS) Test [SP05, pp. 301-303]

Given a controller K ,

$$\|W_M(s)T(s)\|_\infty < 1$$

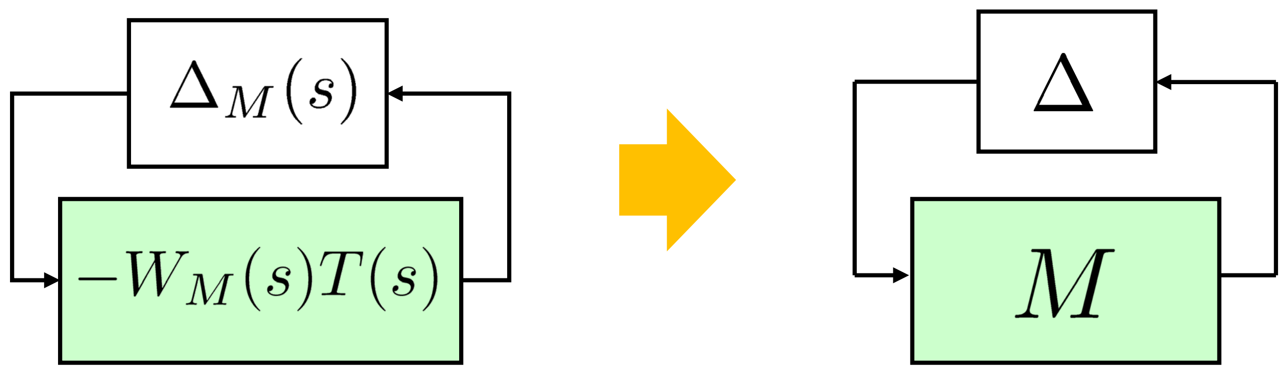
($\because$ Small Gain Theorem)

Again!

H_∞ norm

Small Gain Theorem [SP05, pp. 155, 306]

$\|\Delta_M\|_\infty \leq 1$ $\qquad\qquad$ $\|\Delta\|_\infty \leq 1$



$M\Delta$-structure

The closed-loop system is *internally stable* if $M\Delta$ is stable and satisfies $\|M\Delta\|_\infty < 1$

Multiplicative Property of H_∞ norm (System Gain)

$$\|M\Delta\|_\infty \leq \|M\|_\infty \|\Delta\|_\infty, \|\Delta\|_\infty \leq 1$$

$$\|\Delta_M\|_\infty \leq 1 \text{ and } \|W_M(s)T(s)\|_\infty < 1$$

H_2 norm

$$\|M\Delta\|_2 \not\leq \|M\|_2 \|\Delta\|_2$$

Robust Stability Test in SISO Systems [SP05, p. 277]

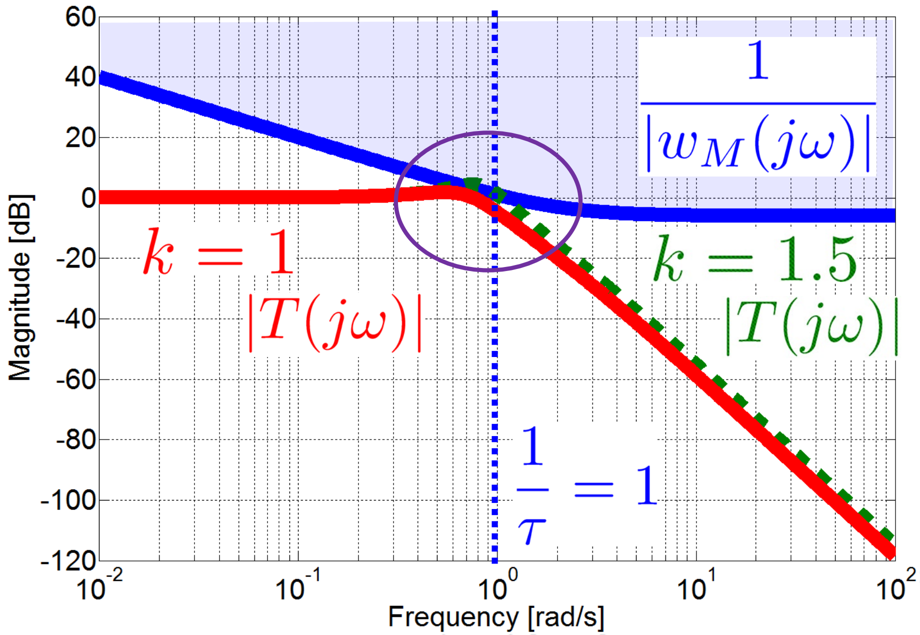
$$\text{(RS)} \quad |T(j\omega)| < \frac{1}{|w_M(j\omega)|}, \ \forall \omega$$

[Ex.] **Perturbed Plant Model**

$$\tilde{P}(s) = e^{-\theta s} P(s), \ 0 \le \theta \le 1$$

Nominal Plant Model

$$P(s) = \frac{3}{(s+1)(5s+1)(10s+1)}$$



Uncertainty Weight

$$w_M(s) = \frac{2s}{s+2} \quad \begin{bmatrix} 1/\tau = 1 \\ r_\infty = 2 \end{bmatrix}$$

Controller

$$K(s) = k\frac{(s+0.2)(10s+1)}{s(0.5s+1)}$$

$k = 1$  RS $(k < 1.28)$

$k = 1.5$  Not RS

Robust Stability Test in SISO Systems

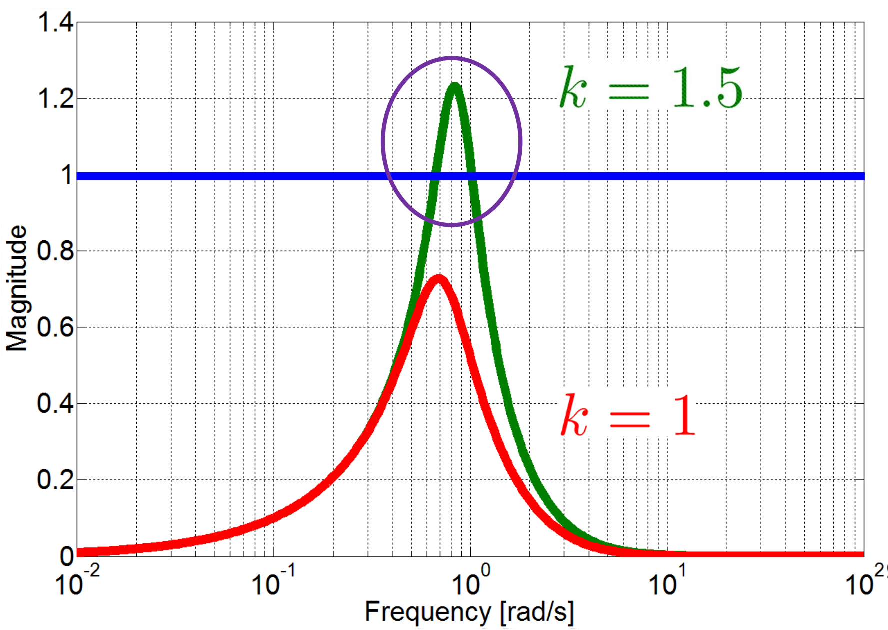
(RS) $\|w_M(s)T(s)\|_\infty < 1$

[Ex.] (Cont.)

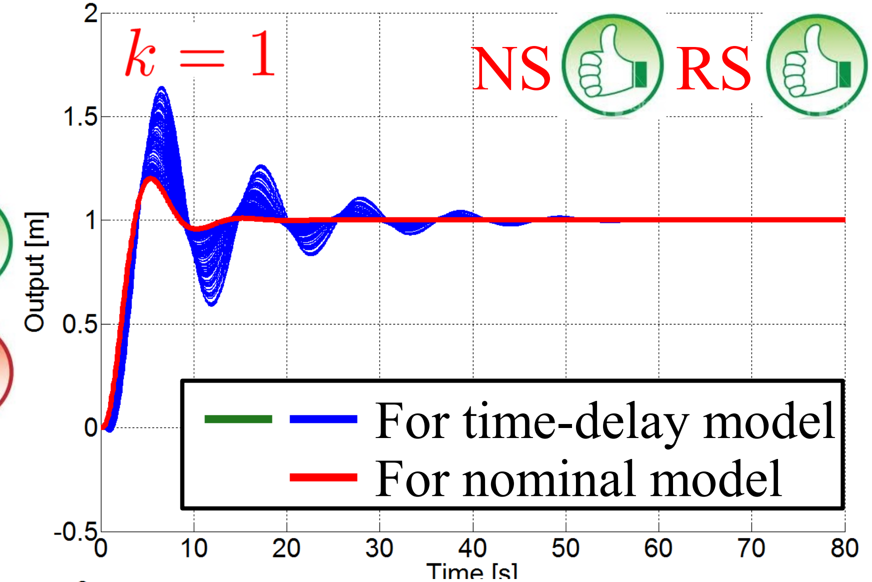
$k = 1 \quad \|w_M(s)T(s)\|_\infty = 0.73$

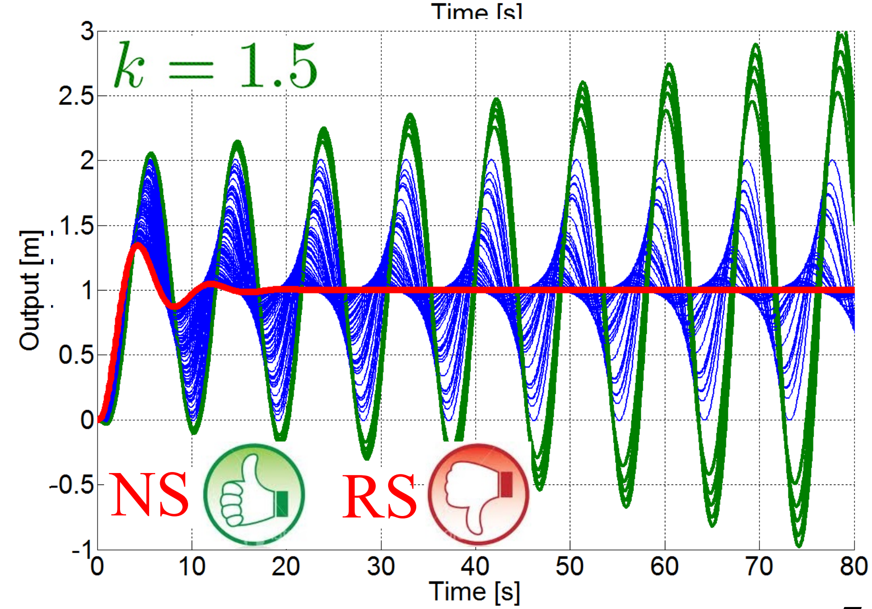
$k = 1.5 \quad \|w_M(s)T(s)\|_\infty = 1.23$

$\bar{\sigma}(w_M(j\omega)T(j\omega))$



Step Responses

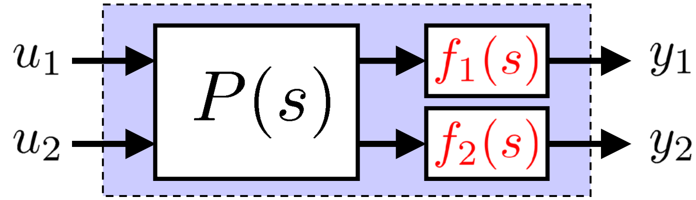




[Ex.] Spinning Satellite: Robust Stability

Uncertain Plant Model (Real System)

$$\tilde{P}(s) = \begin{bmatrix} f_1(s) & 0 \\ 0 & f_2(s) \end{bmatrix} P(s)$$



Nominal Model: $P(s) = \begin{bmatrix} \frac{s-100}{s^2+100} & \frac{10s+10}{s^2+100} \\ \frac{-10s-10}{s^2+100} & \frac{s-100}{s^2+100} \end{bmatrix}$

$$f_i(s) = k_i \frac{-\frac{\theta_i}{2}s+1}{\frac{\theta_i}{2}s+1}, \quad i = 1, 2$$

($\pm$20%, GM = 2dB)

Gain Margin: $0.8 \le k_i \le 1.2$

Delay Margin: $0 \le \theta_i \le 0.02$

Multiplicative (Output) Uncertainty

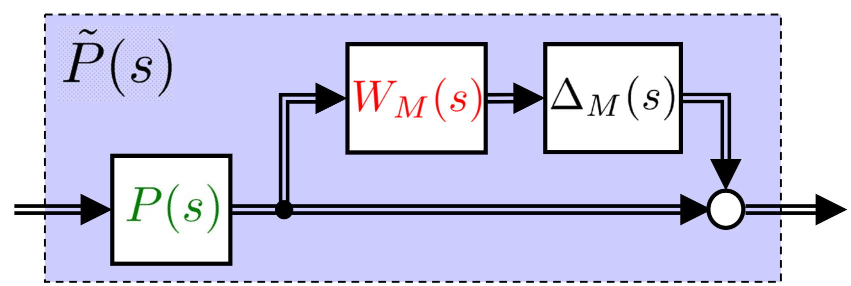
$$\Pi_0 = \{\tilde{P}(s) \,|\, \tilde{P}(s) = (I + \Delta_M(s) W_M(s)) P(s), \ \|\Delta_M\|_\infty \le 1\}$$

$$W_M(s) = w_M(s) I_2,$$

$$w_M(s) = \frac{0.021s + 0.2}{0.0091s + 1}$$

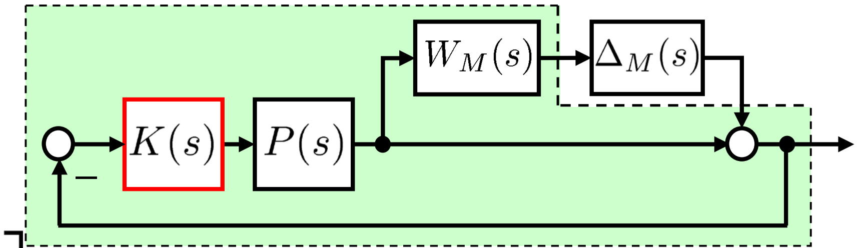
$(\tau = 0.021, \ r_0 = 0.2, \ r_\infty = 2.3)$

$(1/\tau = 48)$



[Ex.] Spinning Satellite: Robust Stability

(RS) $\|W_M(s)T(s)\|_\infty < 1$



Inverse-based Controller K_{inv}

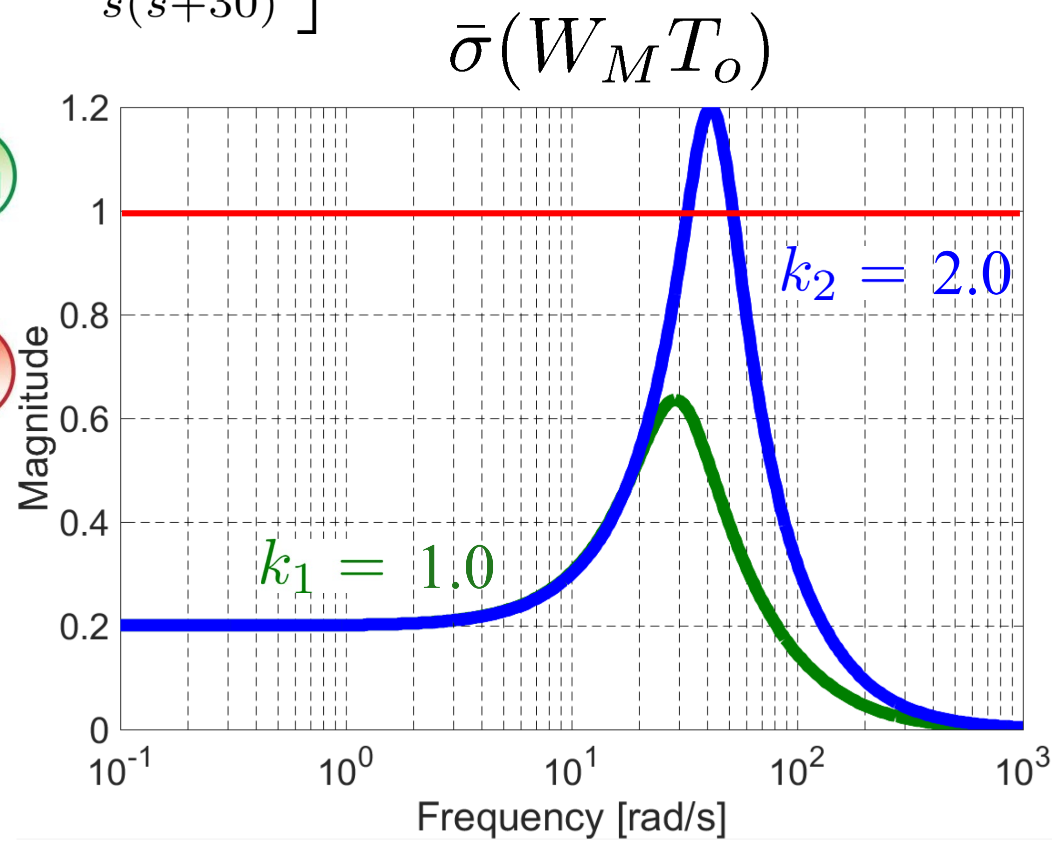
$$K_{\text{inv}}(s) = P^{-1}(s)\begin{bmatrix} \frac{900k}{s(s+30)} & 0 \\ 0 & \frac{900k}{s(s+30)} \end{bmatrix}$$

1) $k_1 = 1.0$

$\|W_M T_o\|_\infty = 0.635$

2) $k_2 = 2.0$

$\|W_M T_o\|_\infty = 1.985$

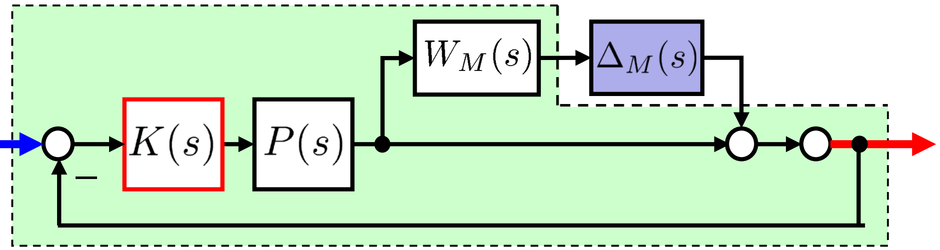


MATLAB Command

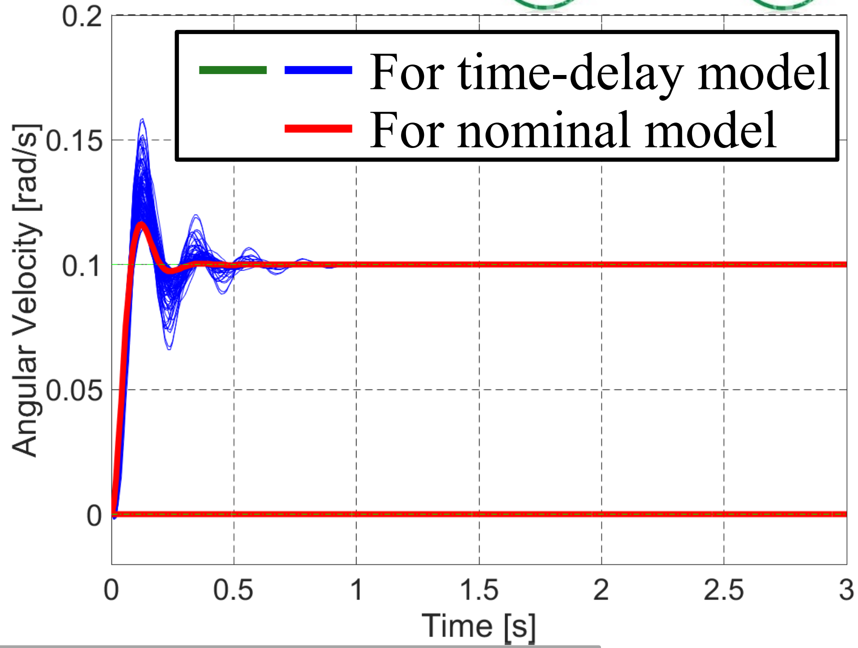
```
[SV,w] = sigma(WM*FI.To) ;
hinfTo = normhinf(WM*FI.To)
%hinfTo = max(max(SV))
figure
semilogx(w,SV)
hold on; grid on;
```

[Ex.] Spinning Satellite: Time Responses for Uncertain Plant

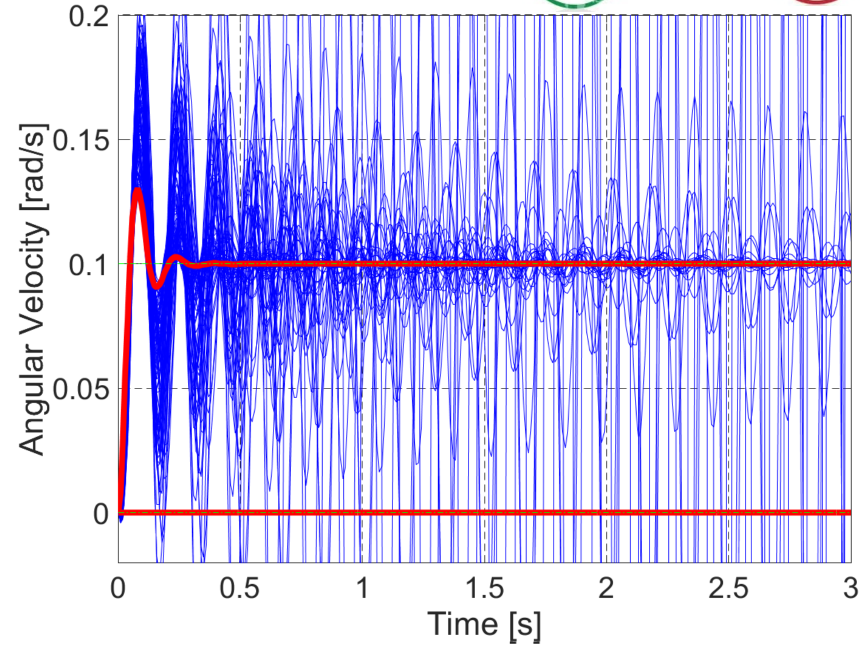
$$w(t) = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} \qquad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$



1) $k_1 = 1.0$ NS RS



2) $k_2 = 2.0$ NS RS

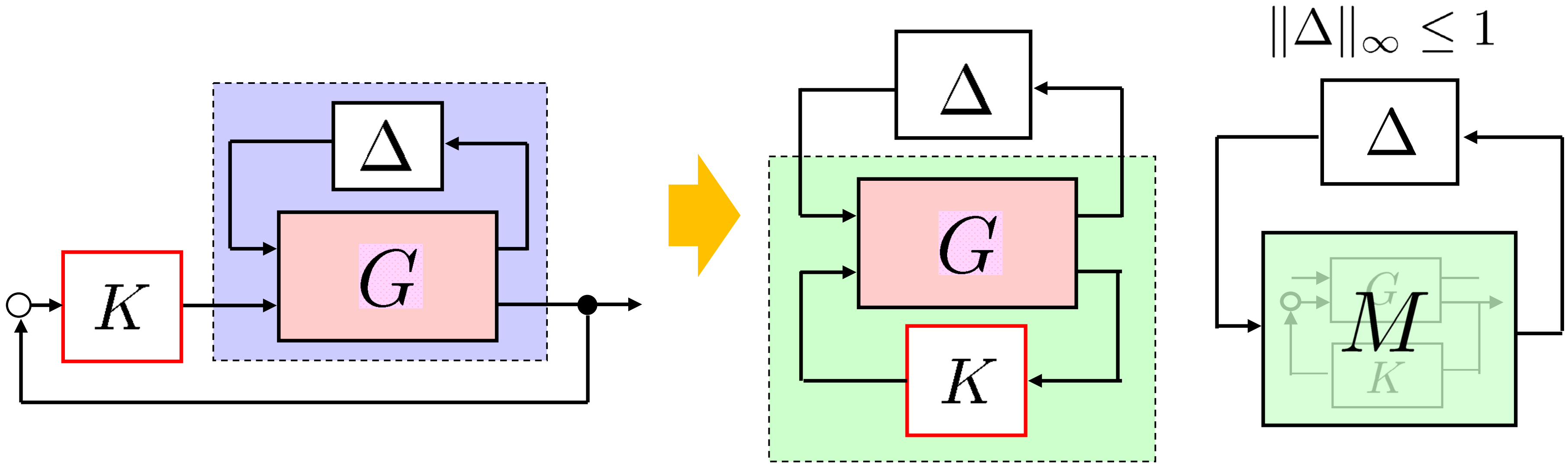


MATLAB Command

```
time = 0:0.01:3;
step_ref = ones(1,length(time));
ref = [0.1*step_ref';
zeros(1,length(time))];
figure; hold on; grid on;
```

```
for i = 1 : 100
    Farray = loopsens(Parray(:,:,i),Kl);  [yhi,t] = lsim(Farray.To,ref,time);
    plot(t,yhi(:,1),'b-'); plot(t,yhi(:,2),'g-');
end
Fl = loopsens(Pnom,Kl);  [yhi,t] = lsim(Fl.To,ref,time);
plot(t,yhi,'r-'); plot(time,ref,'g-.');
```

$M\Delta$-structure and Robust Stability [SP05, pp. 276, 301]



M : Internally stable
(Nominal stability)

Robust Stability (RS) Test

Given a controller K , $\|M\|_\infty < 1$

($\because$ Small Gain Theorem)

Robust Stability Test [SP05, p. 303]

$$\|M\|_\infty < 1, M = W_1 M_0 W_2$$

Unstructured Uncertainty	Perturbed Model Set Π		M_0
Multiplicative (Output)	$(I + W_2 \Delta W_1) P$	Π_1	$PK(I+PK)^{-1} = T_o$
Multiplicative (Input)	$P(I + W_2 \Delta W_1)$	Π_2	$KP(I+KP)^{-1} = T_i$
Inv. Multiplicative (Output)	$(I - W_2 \Delta W_1)^{-1} P$	Π_3	$(I+PK)^{-1} = S_o$
Inv. Multiplicative (Input)	$P(I - W_2 \Delta W_1)^{-1}$	Π_4	$(I+KP)^{-1} = S_i$
Additive	$P + W_2 \Delta W_1$	Π_5	$K(I+PK)^{-1} = KS_o$
Inv. Additive	$P(I - W_2 \Delta W_1 P)^{-1}$	Π_6	$(I+PK)^{-1}P = S_o P$

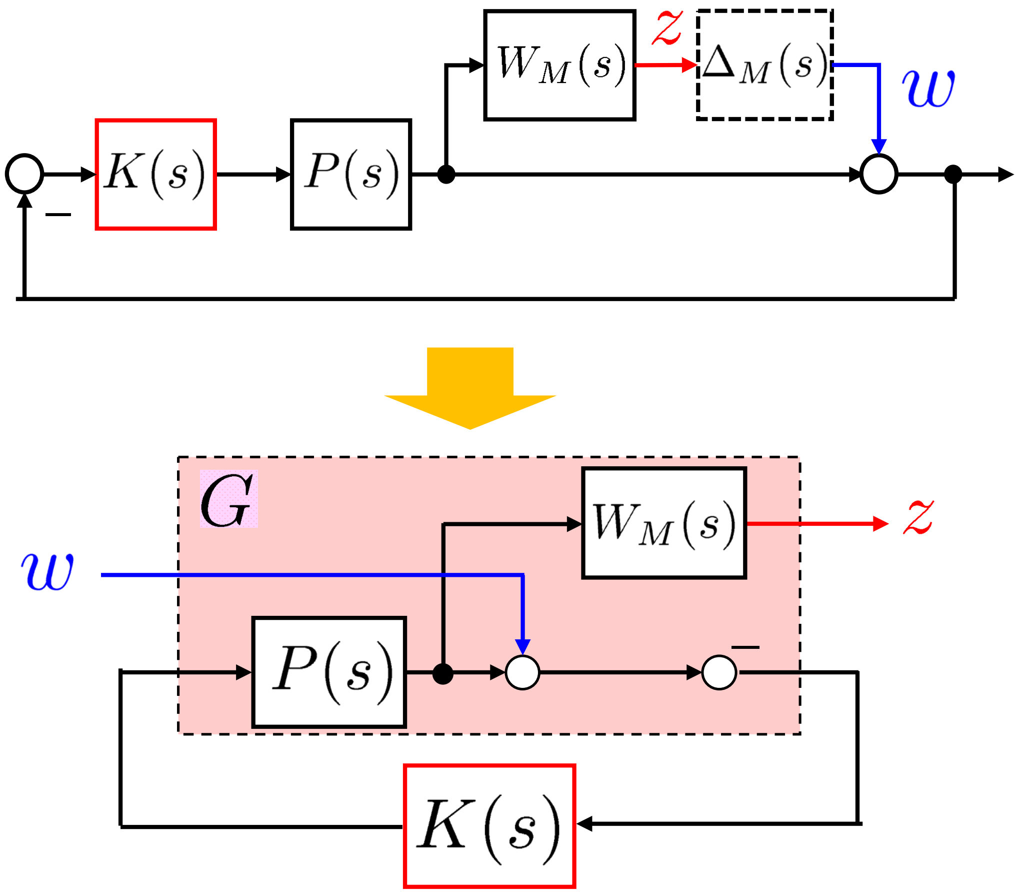
Input Comp. Sens. Func. : $T_i(s) = K(s)P(s)(I + K(s)P(s))^{-1}$

Output Comp. Sens. Func. : $T_o(s) = P(s)K(s)(I + P(s)K(s))^{-1}$

Input Sensitivity Function: $S_i(s) = (I + K(s)P(s))^{-1}$

Output Sensitivity Function: $S_o(s) = (I + P(s)K(s))^{-1}$

Robust Stabilization



$$\|\Delta_M\|_\infty \le 1$$

Again!

LFT

Robust Stabilization Problem

Find all stabilizing controllers K such that

$$\|W_M(s)T(s)\|_\infty < 1$$

Sensitivity Optimization

$$\min_{\text{Feedback } K} \|W_P S\|_\infty = \min_{K} \|W_P (I + PK)^{-1}\|_\infty$$

$$= \min_{Q} \|W_P (I - PQ)\|_\infty \quad (S = I - PQ)$$

(Q -param.)

$$\|W_P S\|_\infty < \gamma$$

Robust Stabilization

$$\min_{\text{Feedback } K} \|W_M T\|_\infty = \min_{K} \|W_M PK (I + PK)^{-1}\|_\infty$$

$$= \min_{Q} \|W_M PQ\|_\infty \quad (T = PQ)$$

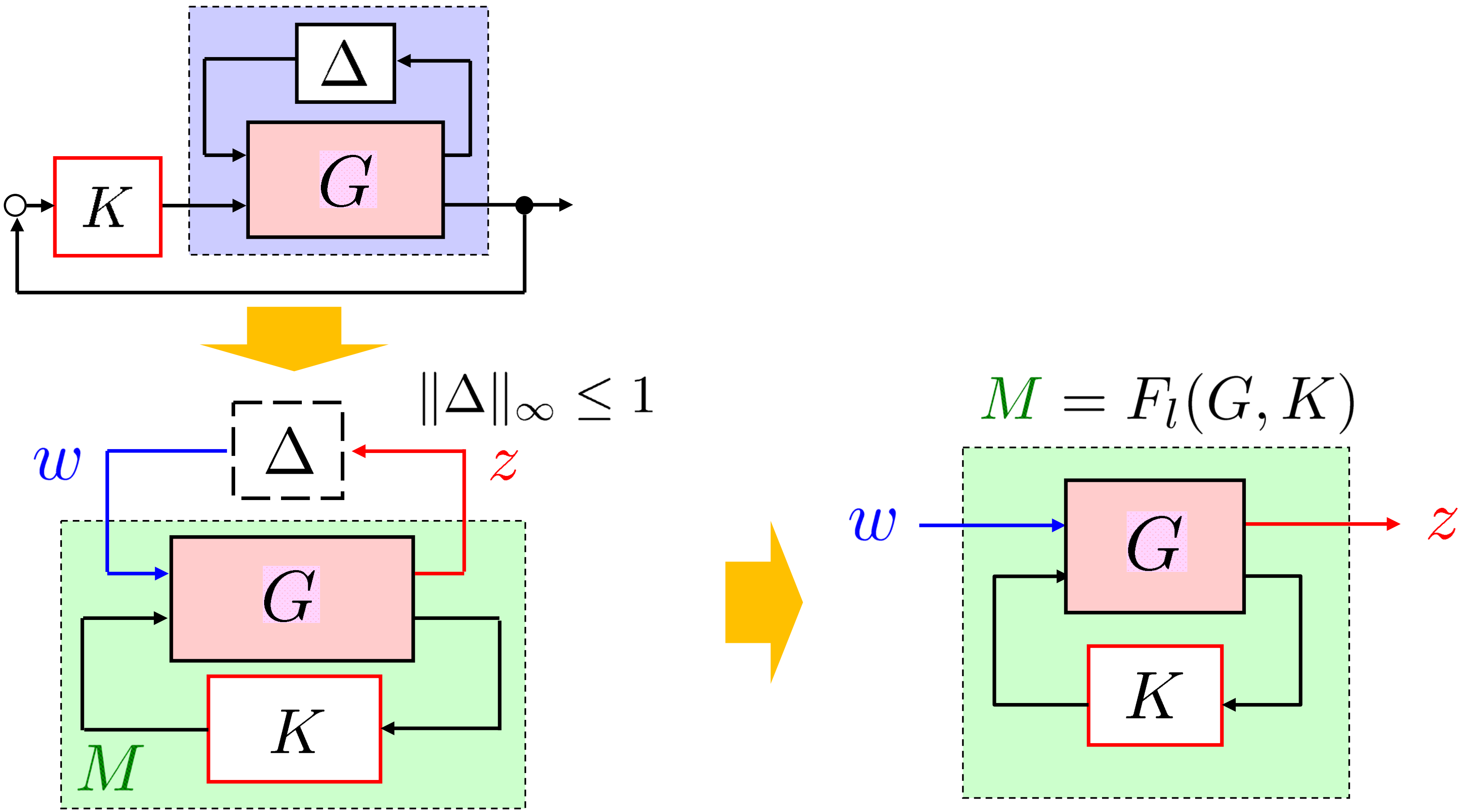
(Q -param.)

Generalization to Unstable Plant

$$\|W_M T\|_\infty = \gamma^* < 1$$

CHECK!

Robust Stabilization



$\Pi \subset \{\Pi_1, \cdots, \Pi_6\}$

H_∞ Control

Robust Stabilization Problem

Find all stabilizing controllers K such that

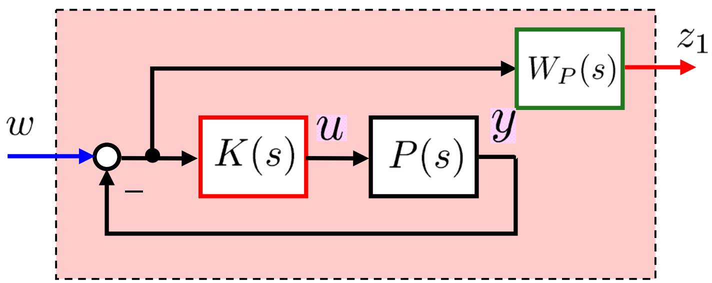
$$\|F_l(G,K)\|_\infty < 1$$

Mixed Sensitivity

G. Zames B. Francis M.Athans M. Safonov

Nominal Performance (NP)

$$\|W_P S\|_\infty < 1$$

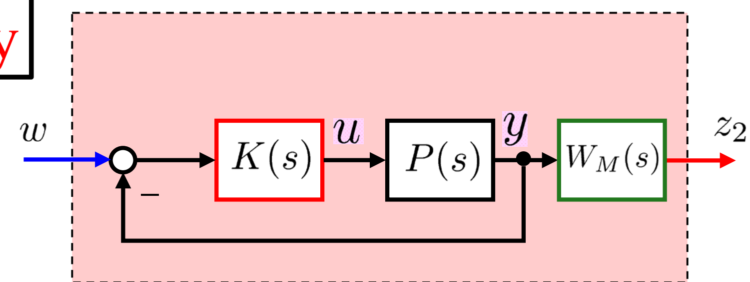


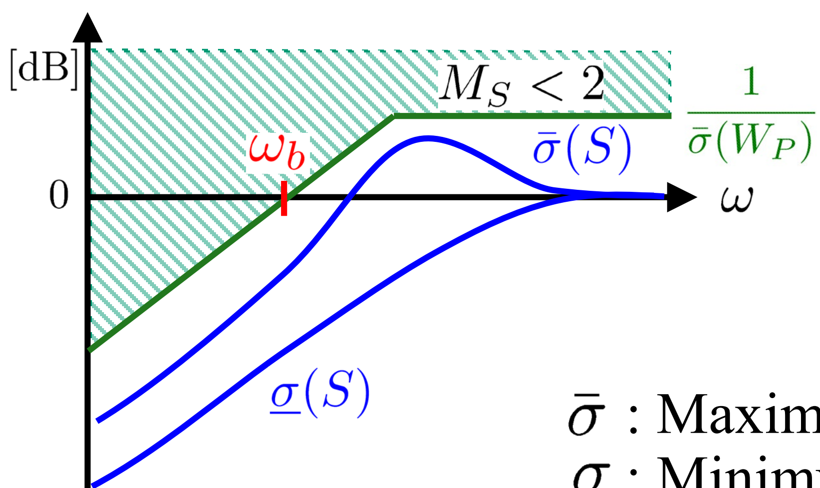
Duality & Complementary

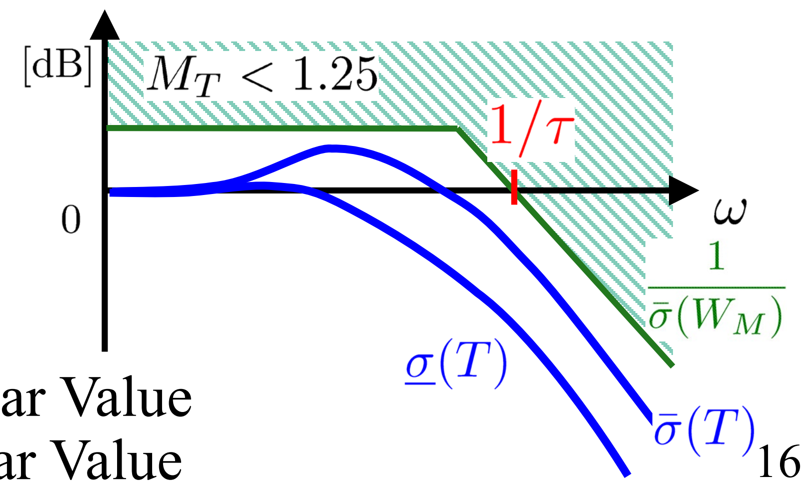
in Robust Control

Robust Stability (RS)

$$\|W_M T\|_\infty < 1$$



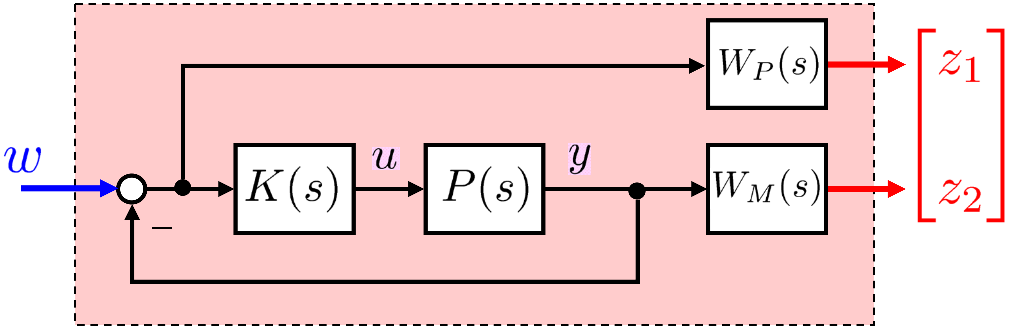




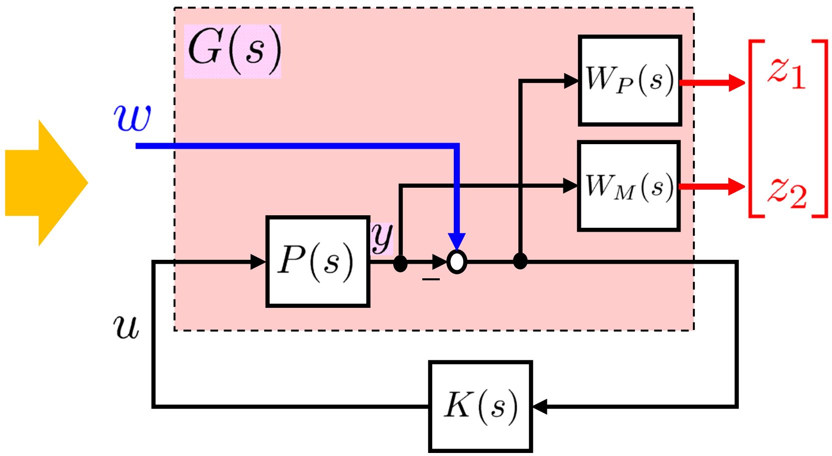
$\bar{\sigma}$: Maximum Singular Value

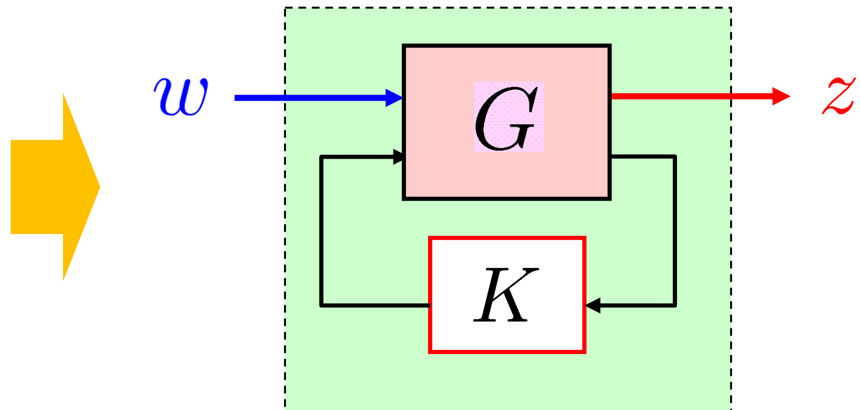
$\underline{\sigma}$: Minimum Singular Value

Mixed Sensitivity: Stacked Requirements [SP05, pp. 62, 282]



$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} W_P(s)S(s) \\ -W_M(s)T(s) \end{bmatrix} w$$



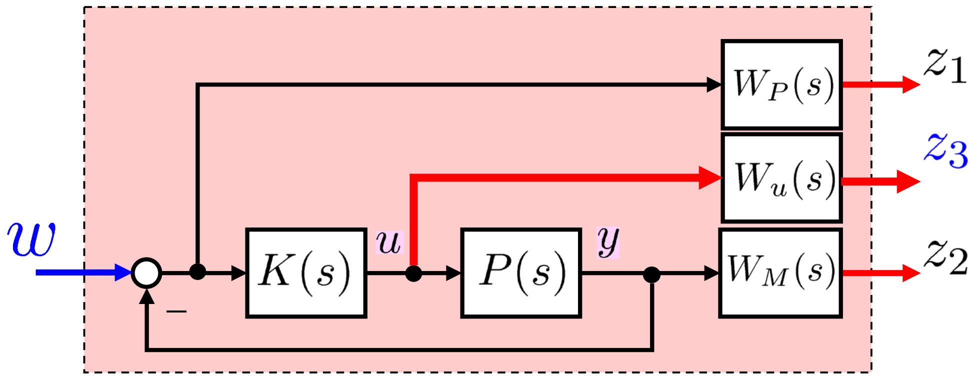


H_∞ Control

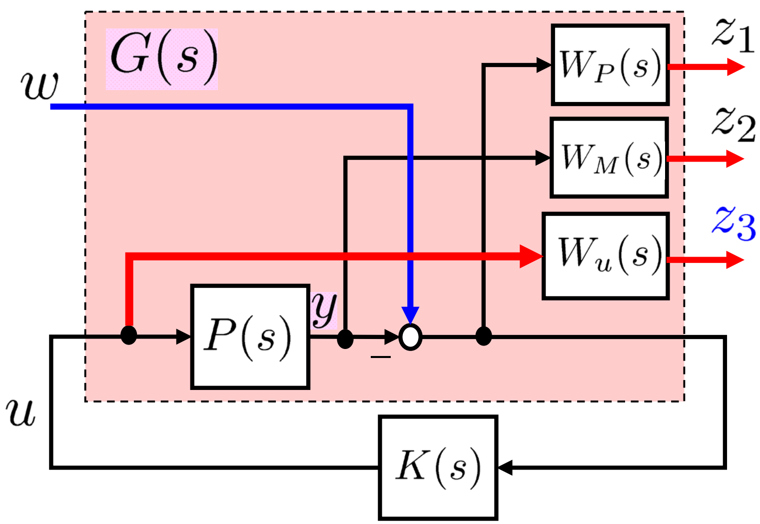
Mixed Sensitivity Problem

$$\text{find } K(s) \text{ s.t. } \left\| \begin{matrix} W_P(s)S(s) \\ W_M(s)T(s) \end{matrix} \right\|_\infty < 1$$

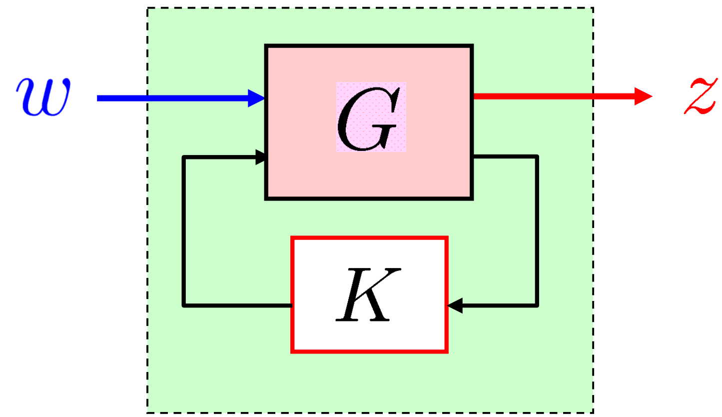
$S/T/KS$ Mixed Sensitivity [SP05, p. 62]



$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} W_P(s)S(s) \\ W_M(s)T(s) \\ W_u(s)K(s)S(s) \end{bmatrix} w$$



Stacked $S/T/KS$ Problem

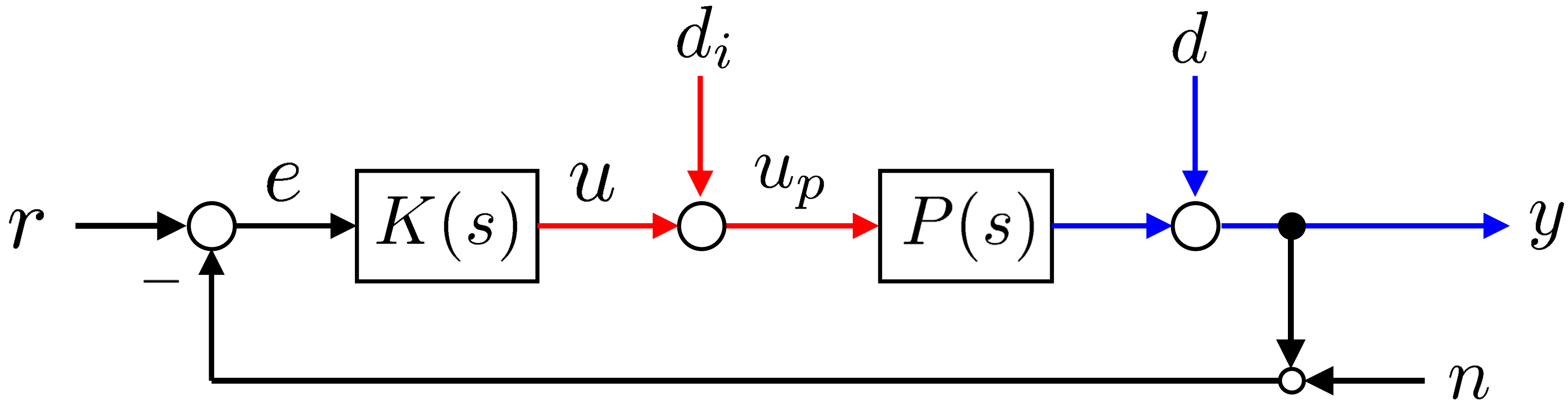


H_∞ Control

$$\text{find } K(s) \text{ s.t. } \left\| \begin{matrix} W_P S \\ W_M T \\ W_u K S \end{matrix} \right\|_\infty < 1$$

Multivariable Loop Shaping [SP05, pp. 341-344]

Loop Transfer Function [SP05, p. 69]



Loop Transfer Function at the *input* to the plant

$$L_i(s) = K(s)P(s)$$

Loop Transfer Function at the *output* to the plant

$$L_o(s) = P(s)K(s)$$

Input Sensitivity Function:

$$S_i(s) = (I + L_i(s))^{-1}$$

Input Comp. Sens. Function:

$$T_i(s) = L_i(s)(I + L_i(s))^{-1}$$

Output Sensitivity Function:

$$S_o(s) = (I + L_o(s))^{-1}$$

Output Comp. Sens. Function:

$$T_o(s) = L_o(s)(I + L_o(s))^{-1}$$

$$L \equiv L_o, \ S \equiv S_o, \ T \equiv T_o$$

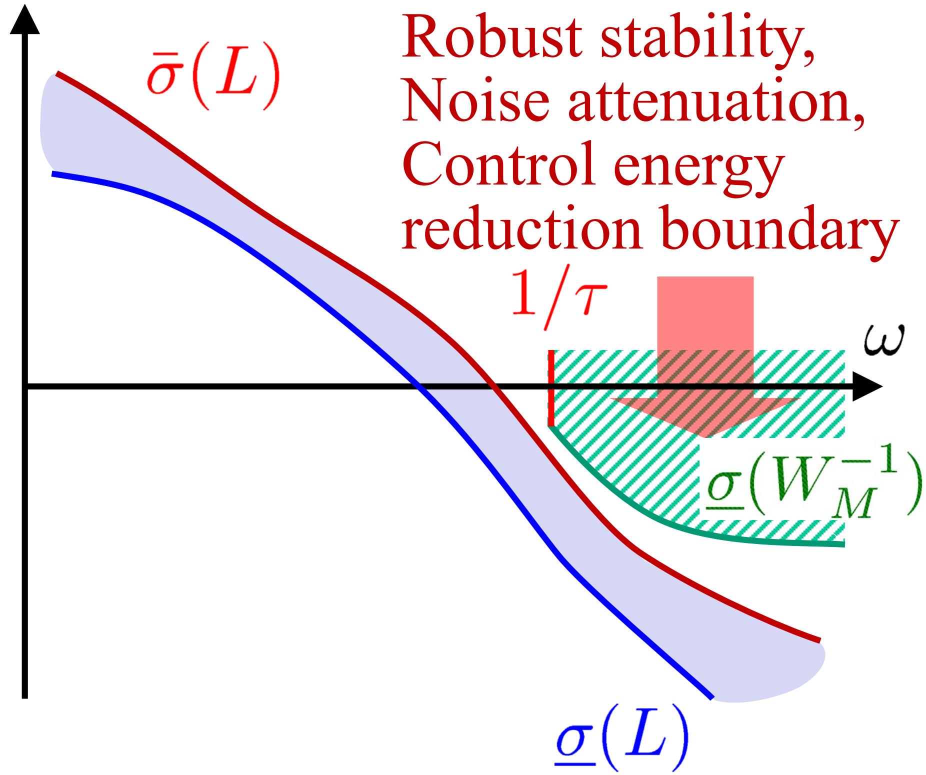
For High Frequencies

$$T = L(I+L)^{-1}$$

If $\bar{\sigma}(L) \ll 1$, $\bar{\sigma}(L) \approx \bar{\sigma}(T)$

(RS) $\bar{\sigma}(T) < \dfrac{1}{\bar{\sigma}(W_M)} = \underline{\sigma}(W_M^{-1})$

➡ $\bar{\sigma}(L) < \underline{\sigma}(W_M^{-1})$, if $\bar{\sigma}(L) \ll 1$



Open/Closed-loop Objectives

$\bar{\sigma}(L) \ll 1 \rightarrow$

Noise attenuation: $\bar{\sigma}(T)$, $\bar{\sigma}(L)$ Small

Input usage (control energy) reduction: $\bar{\sigma}(KS)$, $\bar{\sigma}(K)$ Small

RS to an additive perturbation: $\bar{\sigma}(KS)$, $\bar{\sigma}(K)$ Small

RS to a multiplicative output perturbation: $\bar{\sigma}(T)$, $\bar{\sigma}(L)$ Small

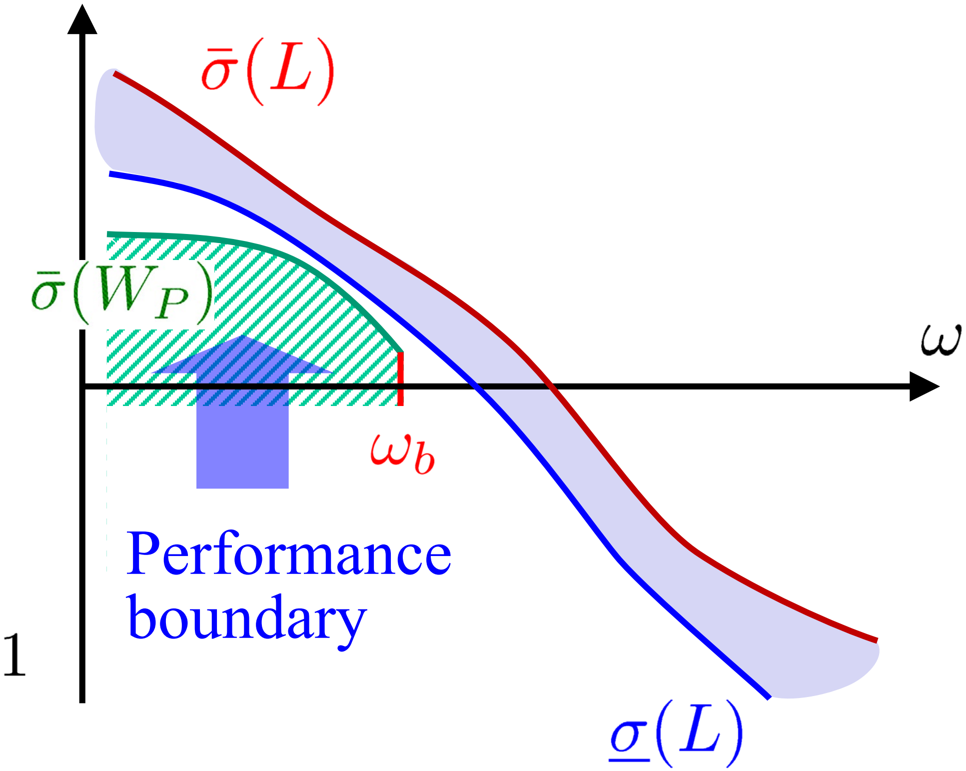
For Low Frequencies

$$\underline{\sigma}(L) - 1 \leq \frac{1}{\bar{\sigma}(S)} \leq \underline{\sigma}(L) + 1$$

If $\underline{\sigma}(L) \gg 1$, $\underline{\sigma}(L) \approx \dfrac{1}{\bar{\sigma}(S)}$

(NP) $\bar{\sigma}(S) < \dfrac{1}{\bar{\sigma}(W_P)}$

➡ $\underline{\sigma}(L) > \bar{\sigma}(W_P)$, if $\underline{\sigma}(L) \gg 1$



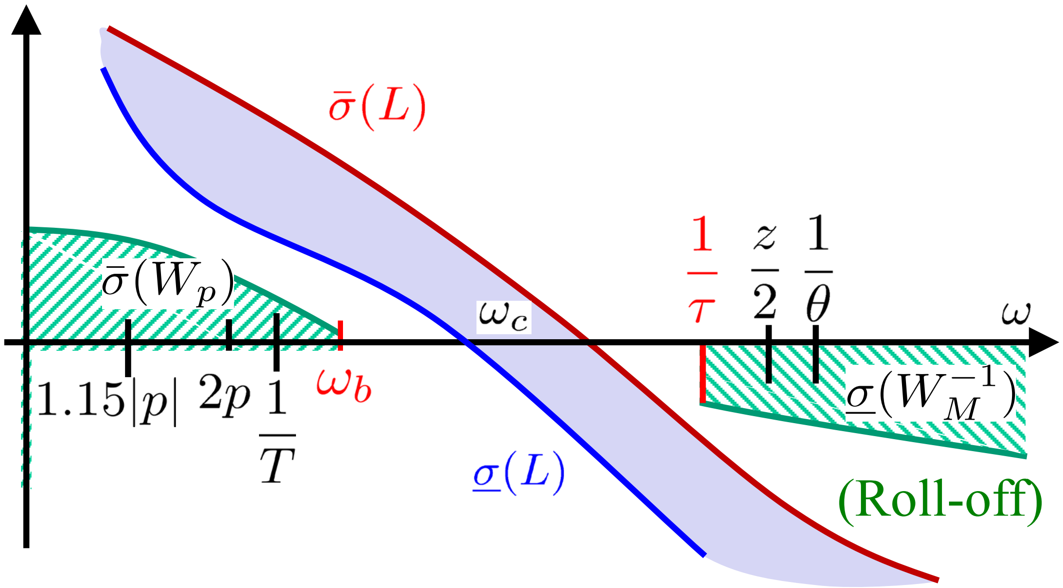
Open/Closed-loop Objectives

$\underline{\sigma}(L) \gg 1 \rightarrow$

Disturbance rejection: $\bar{\sigma}(S)$ Small, $\underline{\sigma}(L)$ Large

Reference Tracking: $\bar{\sigma}(T) \approx \underline{\sigma}(T) \approx 1$, $\underline{\sigma}(L)$ Large

MIMO Loop Shaping [SP05, p. 343]



Performance [Ex.]

p : Unstable Pole

$\frac{1}{T}$: Required Response

Uncertainty [Ex.]

θ : Time Delay

z : Unstable Zero

MIMO Fundamental Limitations

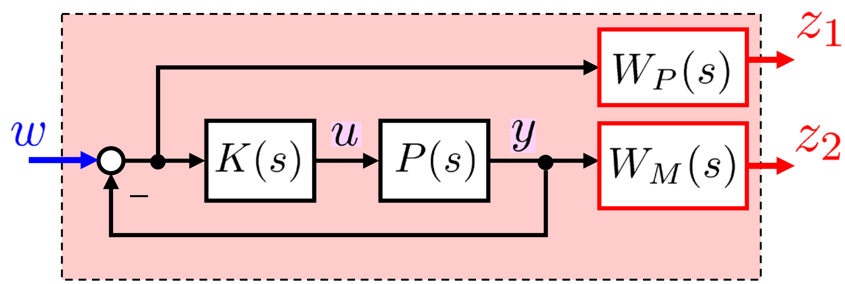
John C. Doyle Gunter Stein

J. C. Doyle and G. Stein, "Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis", IEEE TAC Special Issue on Linear Multivariable Control Systems, 26-1, 1981

[Ex.] Spinning Satellite: Mixed Sensitivity & Loop Shaping

Nominal Model

$$P(s) = \begin{bmatrix} \frac{s-100}{s^2+100} & \frac{10s+10}{s^2+100} \\ \frac{-10s-10}{s^2+100} & \frac{s-100}{s^2+100} \end{bmatrix}$$



Performance Weight

$$W_P(s) = w_p(s) I_2,$$

$$w_p(s) = \frac{0.5s + 11.5}{s + 0.115}$$

$(\omega_b = 11.5, M_s = 2, A = 0.01)$

Uncertainty Weight

$$W_M(s) = w_M(s) I_2,$$

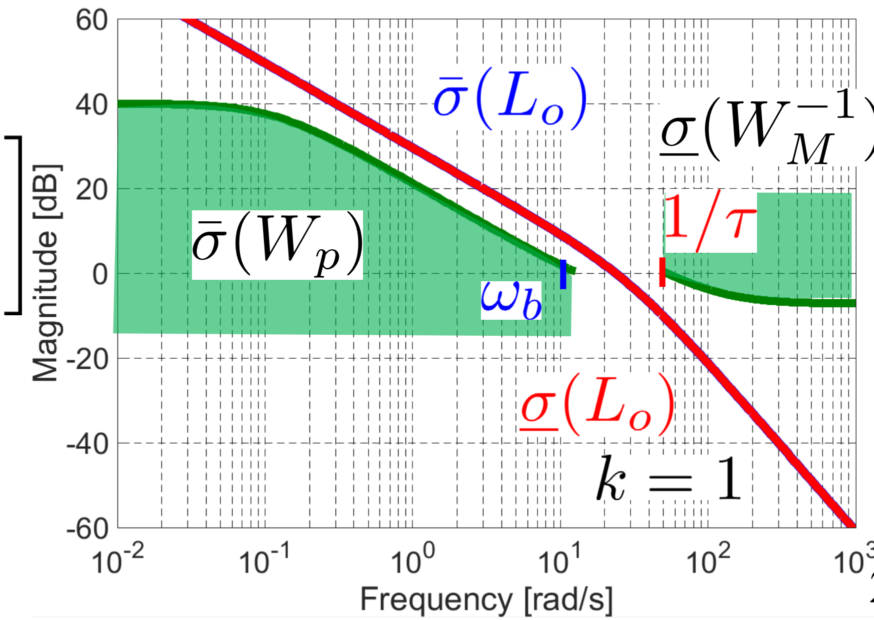
$$w_M(s) = \frac{0.021s + 0.2}{0.0091s + 1}$$

$(1/\tau = 48, \ r_0 = 0.2, \ r_\infty = 2.3)$

Inverse-based Controller:

$$K_{\text{inv}}(s) = P^{-1}(s) \begin{bmatrix} \frac{900k}{s(s+30)} & 0 \\ 0 & \frac{900k}{s(s+30)} \end{bmatrix}$$

NP & RS $0.40 \le k \le 1.64$



MATLAB Command
sigma(FI.Lo,WP,inv(WM))

4. Robust Stability and Loop Shaping

4.1 Robust Stability and Robust Stabilization

[SP05, Sec. 7.5, 8.4, 8.5]

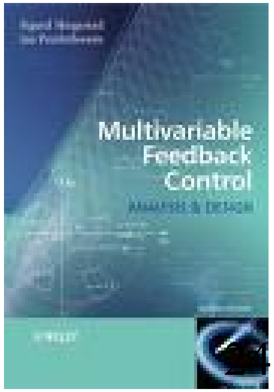
4.2 Mixed Sensitivity and Loop Shaping

[SP05, Sec. 2.6, 2.8, 9.1]

4.3 1st Report

Reference:

[SP05] S. Skogestad and I. Postlethwaite,
Multivariable Feedback Control; Analysis and Design,
Second Edition, Wiley, 2005.



5. H_∞ Control

5.1 General Control Problem Formulation

[SP05, Sec. 3.8]

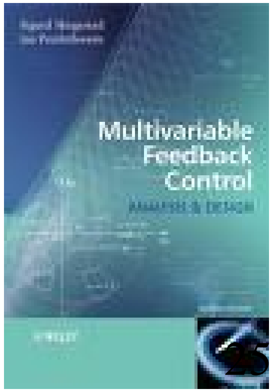
5.2 H_∞ Control Problem and DGKF Solutions

[SP05, Sec. 9.3]

5.3 Structure of H_∞ Controllers

Reference:

[SP05] S. Skogestad and I. Postlethwaite, ***Multivariable Feedback Control; Analysis and Design,*** Second Edition, Wiley, 2005.



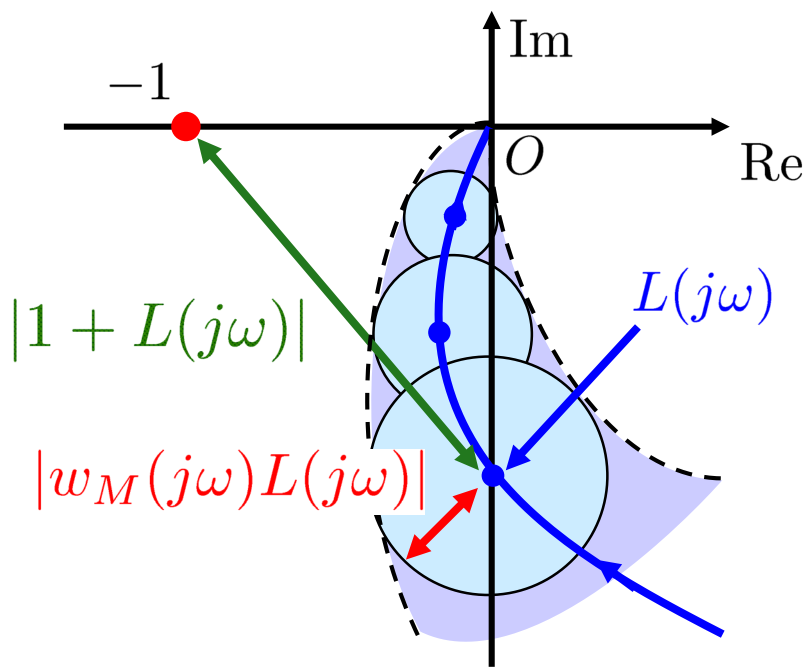
Robust Stability in SISO Systems*

$$\|w_M T\|_\infty < 1 \quad \Leftrightarrow \quad |w_M L| < |1 + L|, \ ^\forall\omega$$

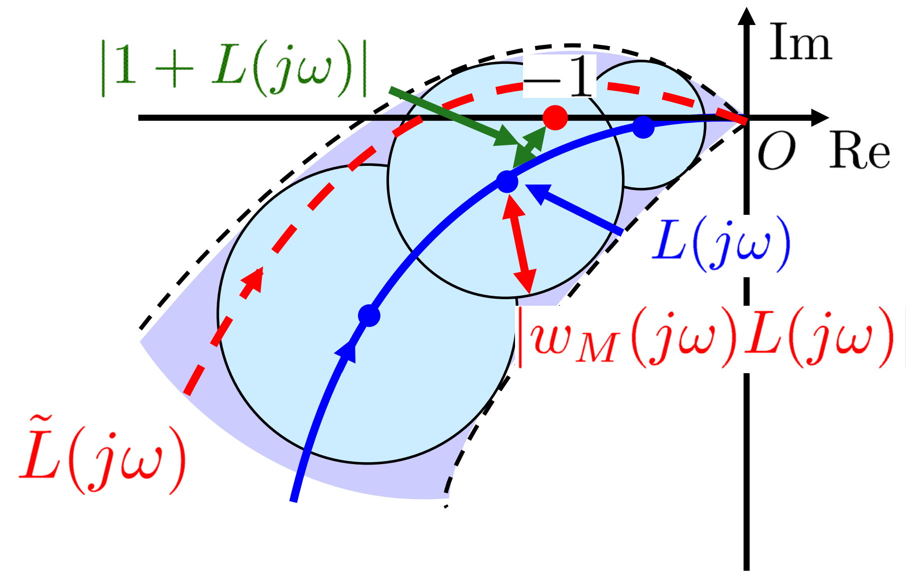
$$\left(T = \frac{L}{1+L} \right)$$

Nyquist Plot [SP05, p. 275]

○ $|w_M L| < |1 + L| \ ^\forall\omega$



× $|w_M L| > |1 + L| \ ^\exists\omega$

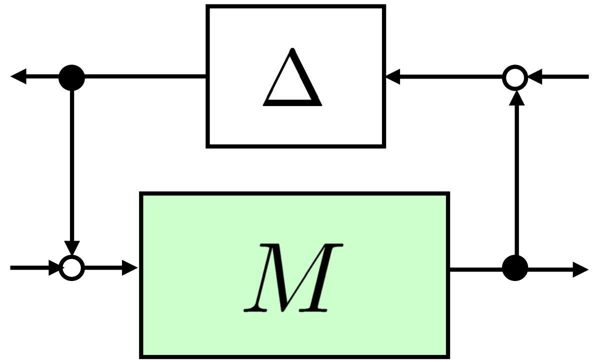


$\tilde{L}$ should not encircle the point -1, $^\forall\tilde{L}$

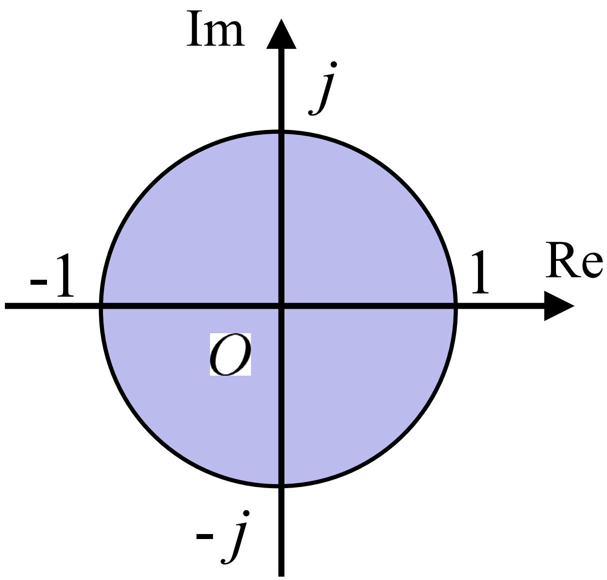
$$\tilde{L} = \tilde{P}K = L + w_M L \Delta_M \quad \|\Delta_M\|_\infty \le 1$$

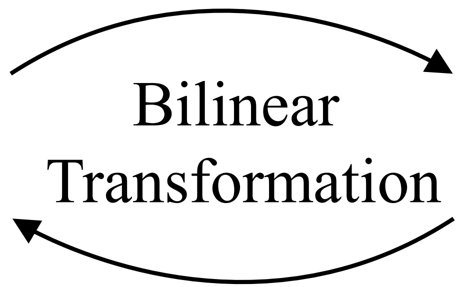
Small Gain Theorem and Passivity Theorem

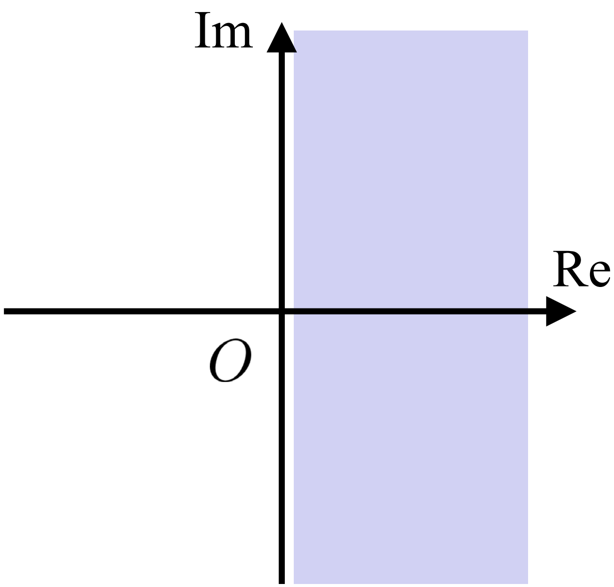
$$\|\Delta\|_\infty \leq 1$$
$$\|M\|_\infty < 1$$



Δ is positive real
M is strictly positive real







Bounded Real
Small Gain Theorem

Positive Real
Passivity Theorem

C. Desoer and M. Videasagar: *Feedback Systems: Input-Output Properties,* Academic Press, 1975

Mixed Sensitivity: Stacked Requirements [SP05, pp. 282-285]

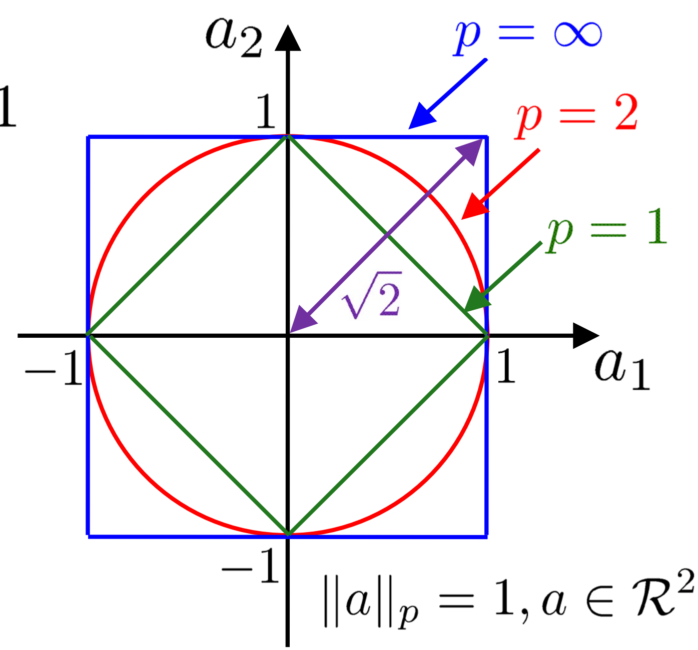
Mixed Sensitivity:

$$\left\| \begin{matrix} W_P(s)S(s) \\ W_M(s)T(s) \end{matrix} \right\|_\infty < 1 \quad \Longleftarrow \quad \begin{cases} \text{NP: } \|W_P S\|_\infty < 1 \\ \text{RS: } \|W_M T\|_\infty < 1 \end{cases}$$

$$\left\| \begin{matrix} w_P S \\ w_M T \end{matrix} \right\|_\infty = \max_\omega \bar{\sigma}\left(\begin{bmatrix} w_P(j\omega)S(j\omega) \\ w_M(j\omega)T(j\omega) \end{bmatrix} \right) < 1$$

$$\max\{\bar{\sigma}(W_P S), \bar{\sigma}(W_M T)\} \le \bar{\sigma}\left(\begin{bmatrix} W_P S \\ W_M T \end{bmatrix} \right)$$

$$\le \sqrt{2} \max\{\bar{\sigma}(W_P S), \bar{\sigma}(W_M T)\}$$



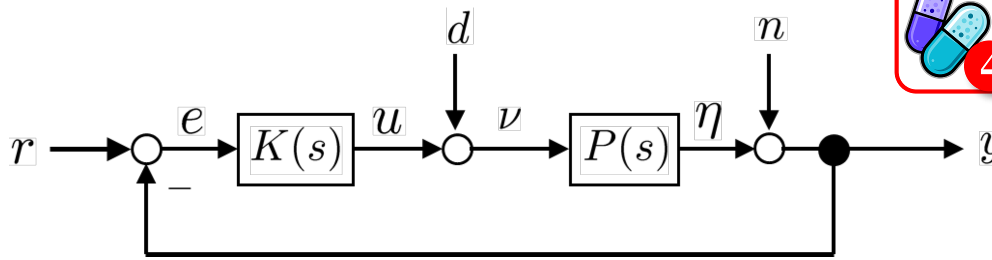
SISO Systems

$$\bar{\sigma}\left(\begin{bmatrix} w_P S \\ w_M T \end{bmatrix} \right) = \sqrt{|w_P S|^2 + |w_M T|^2}$$

SISO Robust Performance

$$|w_P(j\omega)S(j\omega)| + |w_M(j\omega)T(j\omega)| < 1, \ \forall \omega$$

Loop Shaping

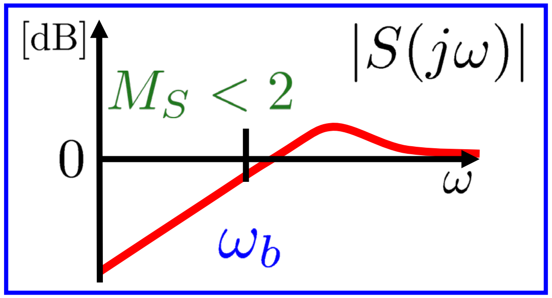


Loop Transfer Function

$$L(s) = P(s)K(s)$$

Sensitivity: $S = \frac{1}{1+L}$

Comp. Sensitivity: $T = \frac{L}{1+L}$



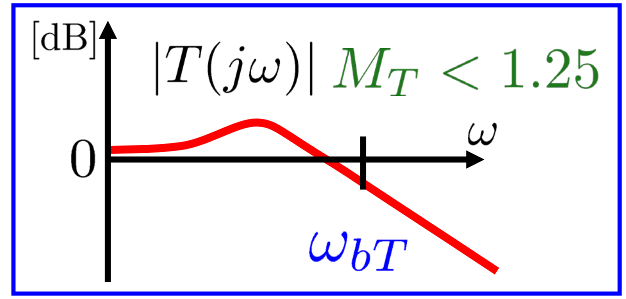
$|L| \gg 1 \rightarrow |S| \ll 1$

large small

+

Constraint

$S + T = 1$



$|L| \ll 1 \rightarrow |T| \ll 1$

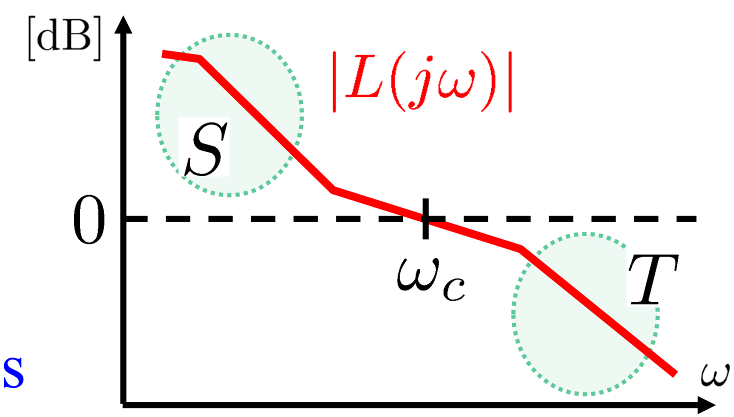
small small

Loop Shaping

Closed-loop S, T

→ Open Loop L

Stability, Performance, Robustness



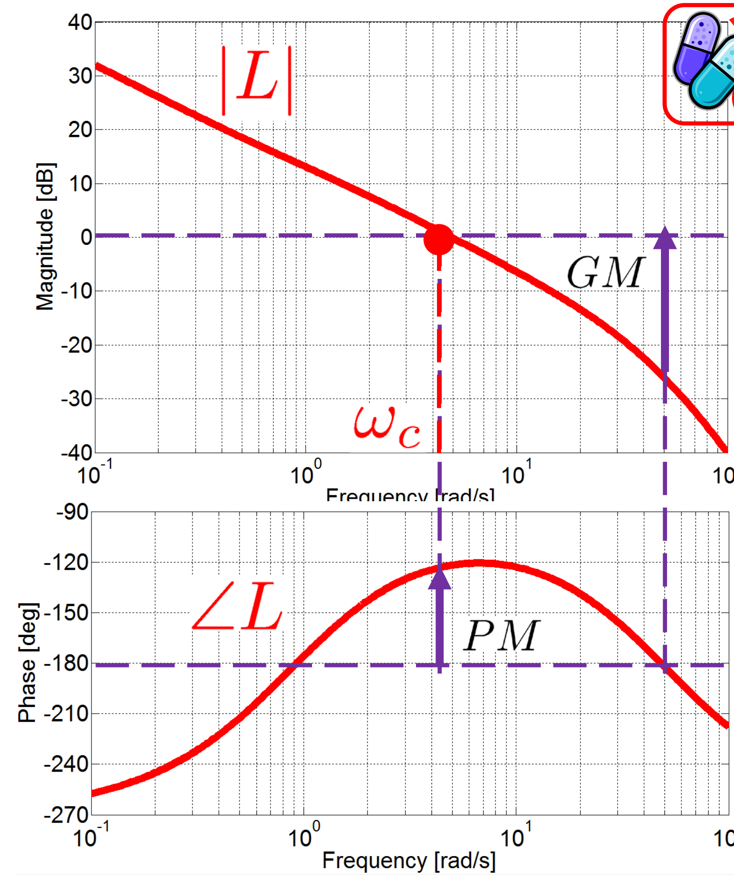
Loop Transfer Function

$$L(s) = P(s)K(s)$$

[SP05, Ex. 2.4] (p. 34)

$$P(s) = \frac{4}{(s-1)(0.02s+1)^2}$$

$$K(s) = 1.25\left(1 + \frac{1}{1.25s}\right)$$



Gain Crossover Frequency

$\omega_c = 4.9$ [rad/s] $\quad |L(j\omega_c)| = 1$

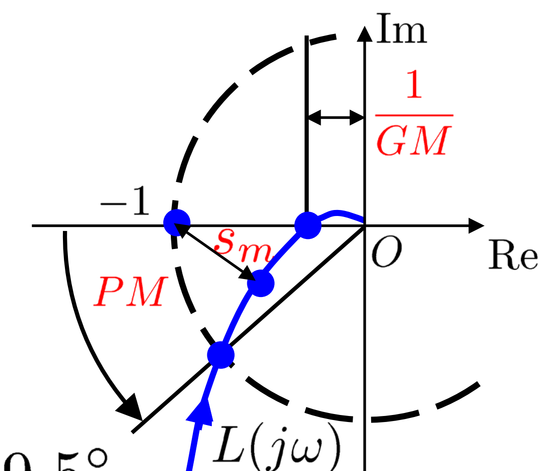
Stability Margins [SP05, p. 32]

Gain Margin $\quad GM$: $2 \sim 5$ ($6 \sim 14$ dB)

Phase Margin $\quad PM$: $30^\circ \sim 60^\circ$

Time Delay Margin $\theta = PM/\omega_c$

Stability Margin $s_m = 1/M_S$: $0.5 \sim 0.8$



[SP05, Ex. 2.4] (p. 34) $GM = 18.7 \quad PM = 59.5^\circ$

Frequency Domain Performance

[SP05, Ex. 2.4] (p. 34)

$M_S = 1.19$ $\quad$ $M_T = 1.38$

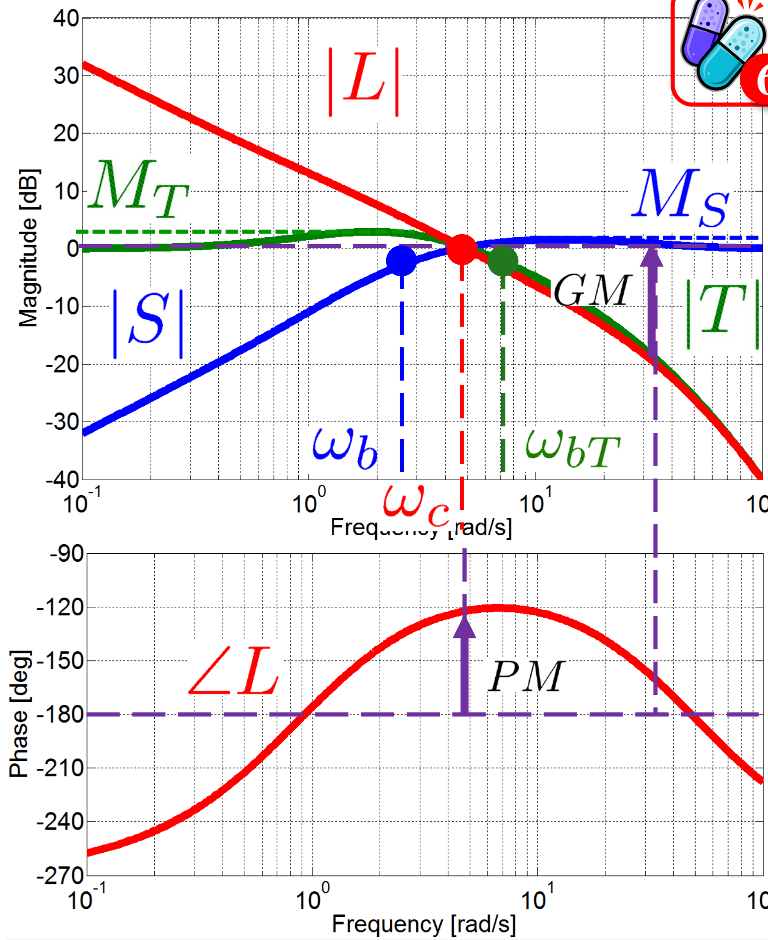
$M_S < 2$ $\quad$ $M_T < 1.25$

$\omega_b = 2.6\ [\mathrm{rad/s}]$ $\quad$ $\omega_{bT} = 7.8\ [\mathrm{rad/s}]$

$\omega_c = 4.9\ [\mathrm{rad/s}]$

$$\omega_b < \omega_c < \omega_{bT} \quad (PM < 90^\circ)$$

$GM = 18.7 \quad PM = 59.5^\circ$



Maximum Peak Criteria [SP05, p. 36]

$$GM \geq \frac{M_S}{M_S - 1}, PM \geq 2\sin^{-1}\left(\frac{1}{2M_S}\right) [\mathrm{rad}]$$

$$GM \geq 1 + \frac{1}{M_T}, PM \geq 2\sin^{-1}\left(\frac{1}{2M_T}\right) [\mathrm{rad}]$$

[Ex.] $M_S = 2$

$\rightarrow GM \geq 2,\ PM \geq 29.0^\circ$

[Ex.] $M_T = 1.25$

$\rightarrow GM \geq 1.8,\ PM \geq 46.0^\circ$

Bode Gain-phase Relationship [SP05, p. 18]

$$\angle G(j\omega_c) \approx \frac{\pi}{2} \underbrace{\left(\frac{d \ln |G(j\omega)|}{d \ln \omega} \right)_{\omega=\omega_c}}_{n_c}$$

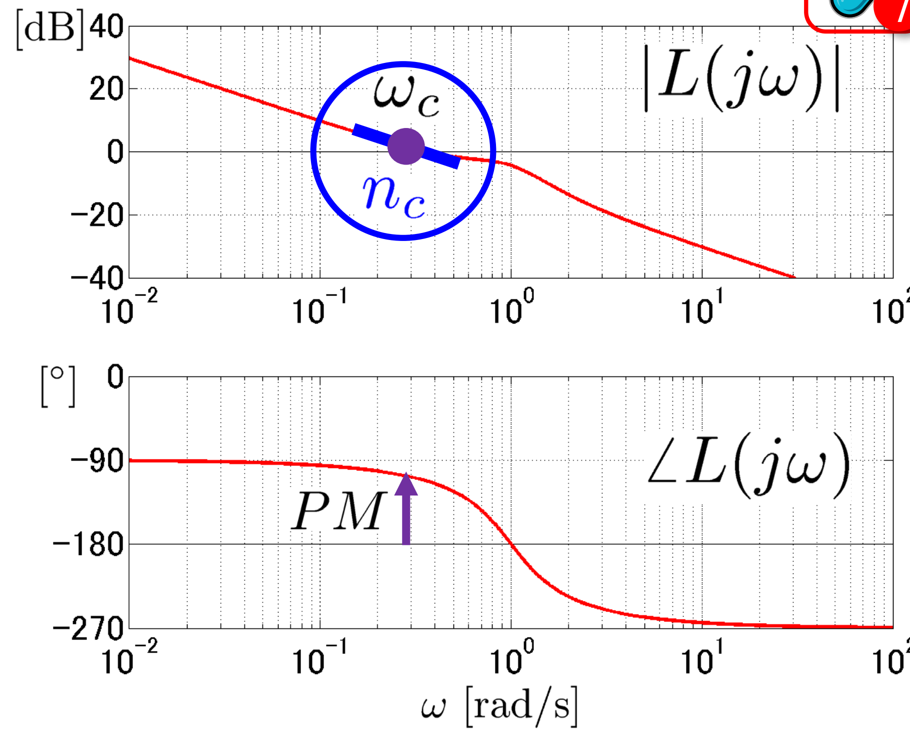
(minimum phase systems)

Slope of the Gain Curve at ω_c

$$n_c = -1 \rightarrow \angle G(j\omega_c) = -90^\circ$$

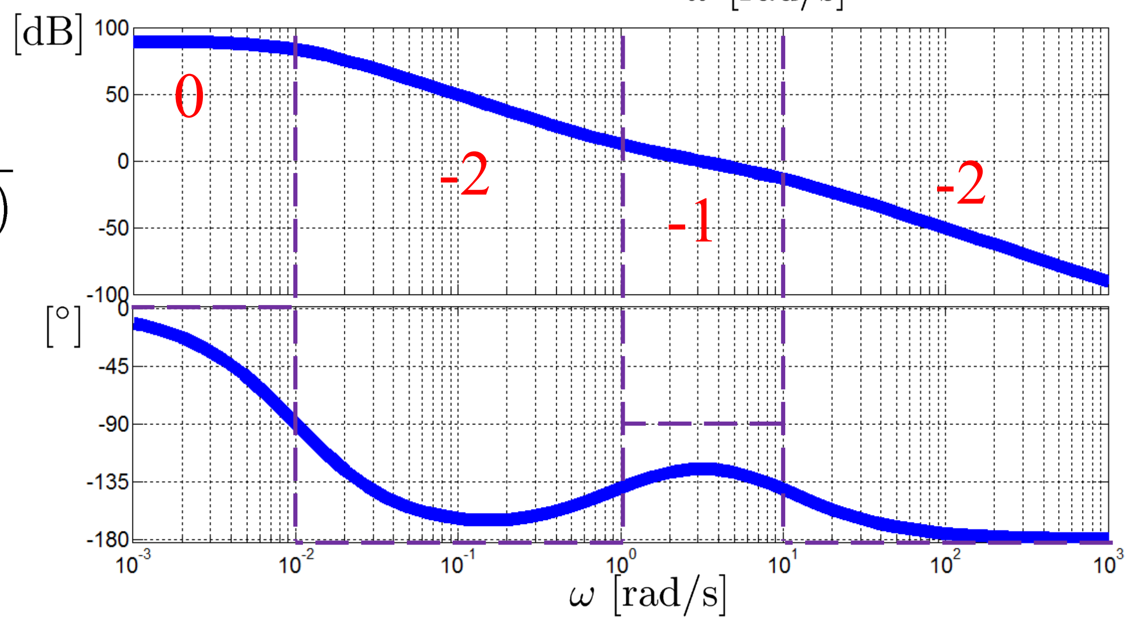
$$n_c = -2 \rightarrow \angle G(j\omega_c) = -180^\circ$$

Steep Slope: Small Phase Margin

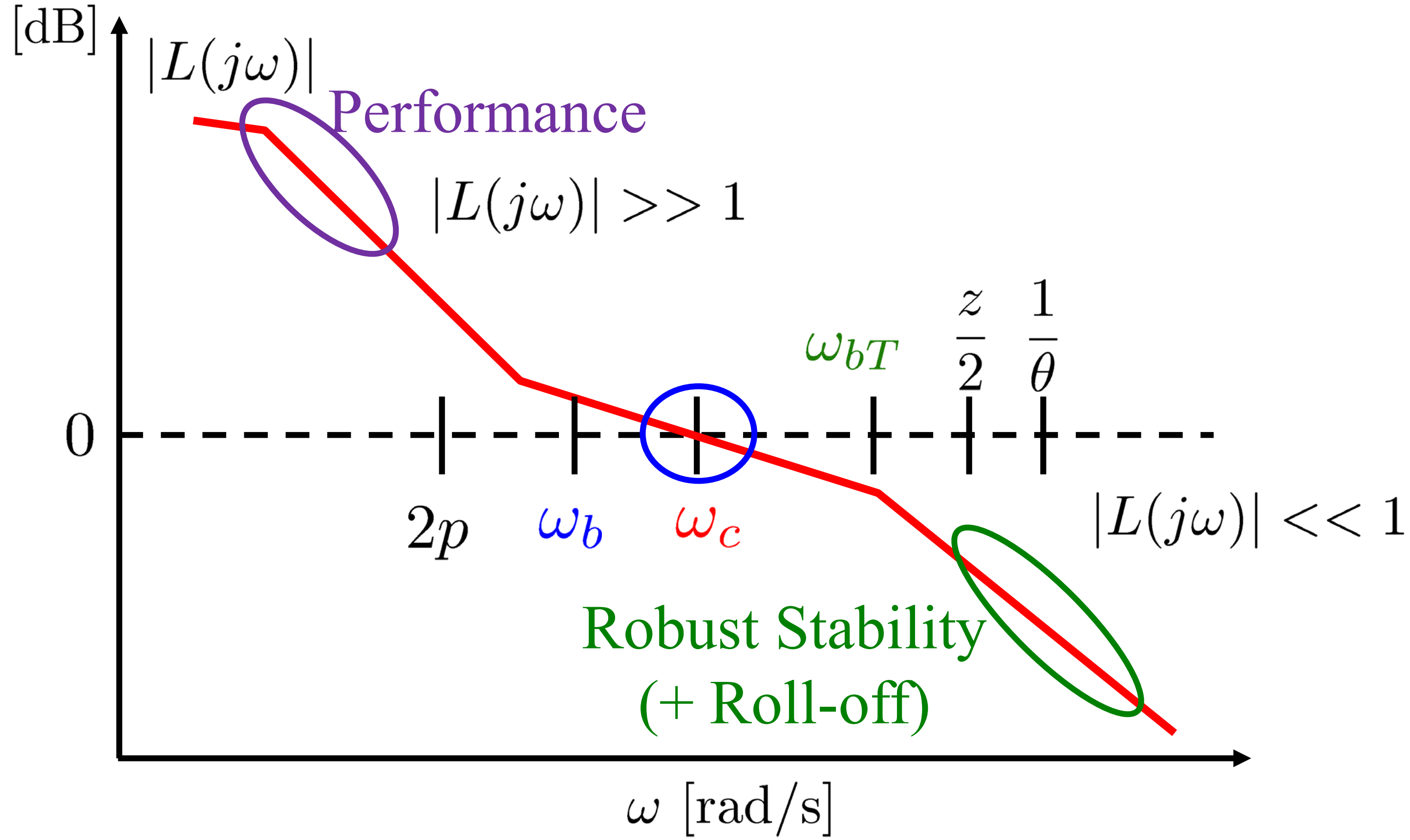


[SP05, Ex., p. 20]

$$L(s) = \frac{30(s+1)}{(s+0.01)^2(s+10)}$$



SISO Loop Shaping [SP05, pp. 41, 42, 343]



Loop Shaping Specifications

- Gain Crossover Frequency ω_c
- Shape of $L(j\omega)$
- System Type, Defined as the Number of Pure Integrators in $L(s)$
- Roll-off at Higher Frequencies