Robust Control

Spring, 2019

Instructor: Prof. Masayuki Fujita (S5-303B)

5th class

Tue., 14th May, 2019, 10:45～12:15,

S423 Lecture Room

5. H_∞ Control

5.1 General Control Problem Formulation

[SP05, Sec. 3.8]

5.2 H_∞ Control Problem and DGKF Solutions

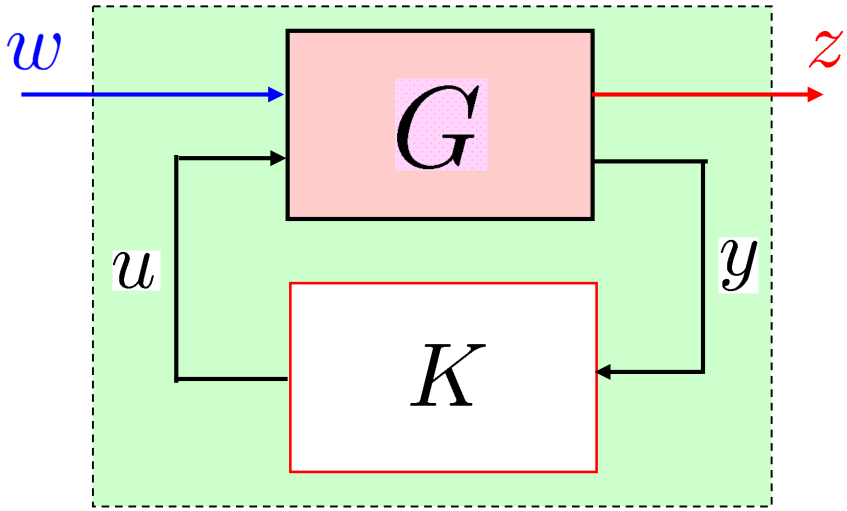
[SP05, Sec. 9.3]

5.3 Structure of H_∞ Controllers

Reference:

[SP05] S. Skogestad and I. Postlethwaite, ***Multivariable Feedback Control; Analysis and Design,*** Second Edition, Wiley, 2005.

General Control Problem Formulation [SP05, p. 104]



u : control inputs

y : measured (or sensor) outputs

w : exogenous inputs (disturbance and commands, etc.)

z : regulated outputs

Generalized Plant

$$\begin{bmatrix} z \\ y \end{bmatrix} = G(s) \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

$$u = K(s)y$$

Closed-loop Transfer Function (LFT)

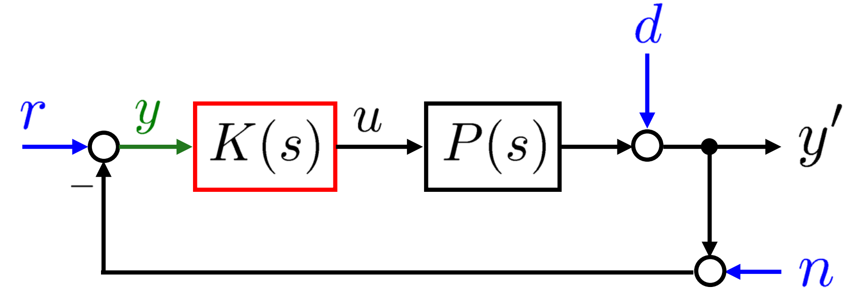
lft(G,K)

$$z = F_l(G,K)w$$

$$F_l(G,K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$$

General Control Problem Formulation

[SP05, Ex. 3.18] (p. 105) Building Interconnection

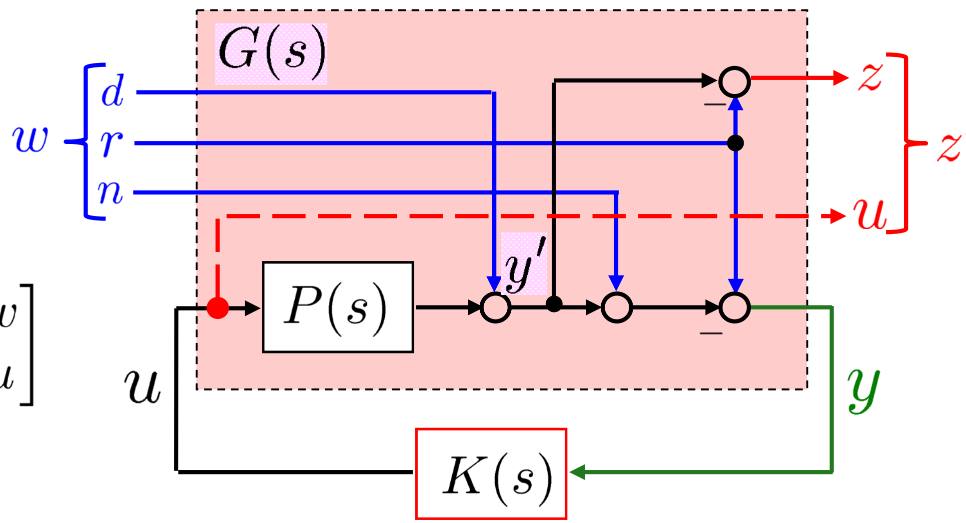


Exogenous Inputs $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} d \\ r \\ n \end{bmatrix}$

Regulated Output $z = y' - r$

Measured Output $y = r - y' - n$

Control Input $u = u$



$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} y' - r \\ r - y' - n \end{bmatrix} = \begin{bmatrix} Pu + d - r \\ r - Pu - d - n \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} I & -I & 0 & P \\ \hline -I & I & -I & -P \end{array}\right] \begin{bmatrix} d \\ r \\ n \\ u \end{bmatrix} = G \begin{bmatrix} w \\ u \end{bmatrix}$$

Generalized Plant

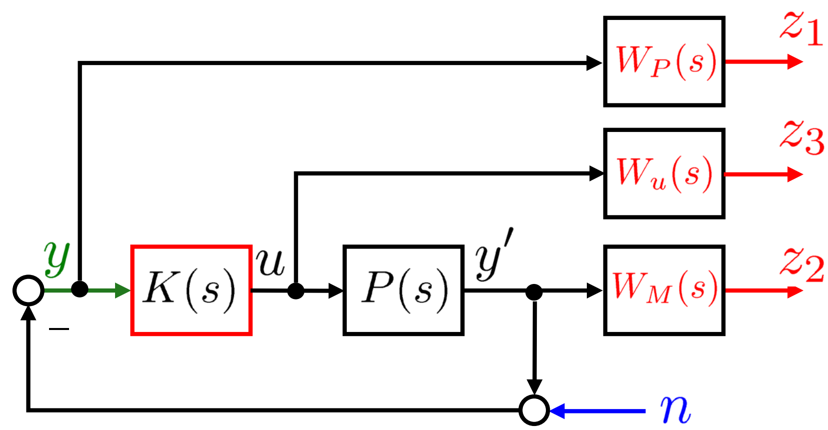
$$G = \left[\begin{array}{ccc|c} I & -I & 0 & P \\ \hline -I & I & -I & -P \end{array}\right]$$

Remark

$$z' = \begin{bmatrix} y' - r \\ u \end{bmatrix}$$

General Control Problem Formulation

[SP05, Ex. 3.19] (p. 107) Including Weights



Exogenous Input $\quad w = n$

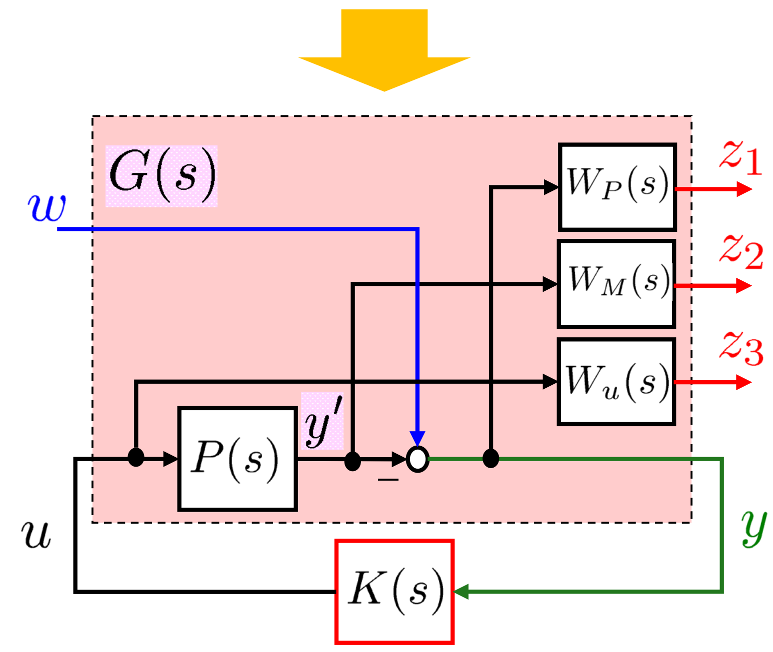
Regulated Outputs $\quad z = \begin{bmatrix} W_P(y'+w) \\ W_M y' \\ W_u u \end{bmatrix}$

Measured Output $\quad y = -y' - n$

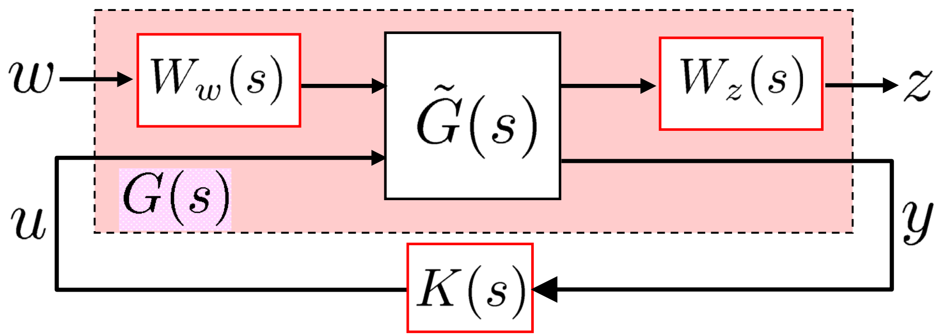
Control Input $\quad u = u$

Generalized Plant

$$\begin{bmatrix} z \\ y \end{bmatrix} = G \begin{bmatrix} w \\ u \end{bmatrix}, \quad G = \left[\begin{array}{c|c} W_P & W_P P \\ 0 & W_M P \\ 0 & W_u \\ \hline -I & -P \end{array}\right]$$



Remark



LQG Type Control Problem Formulation [SP05, pp. 344, 356]

$$\dot{x} = Ax + Bu + w_d$$
$$y = Cx + w_n$$

Regulated Outputs

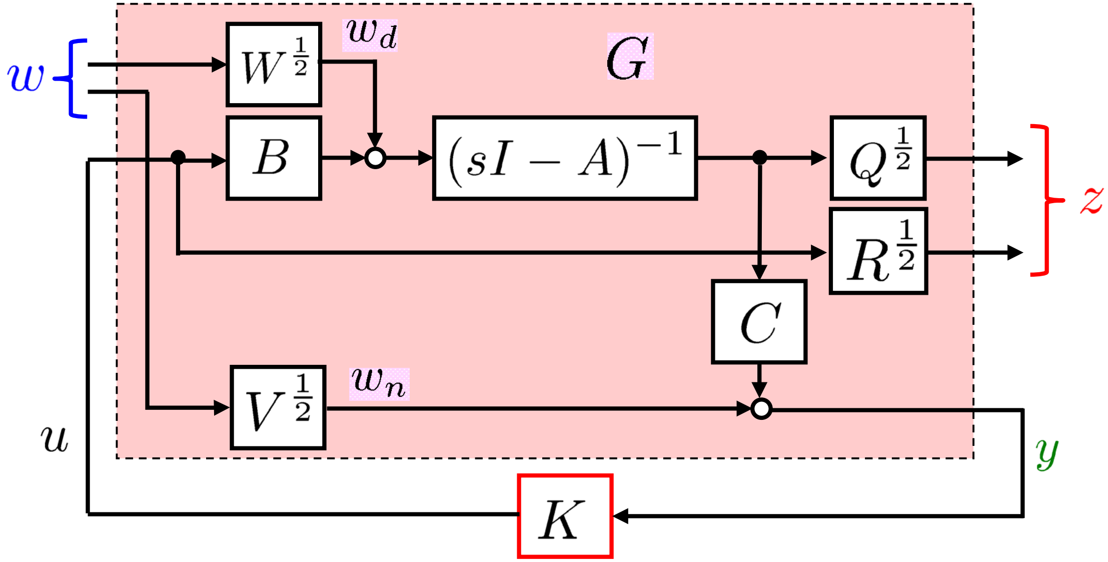
$$z = \begin{bmatrix} Q^{\frac{1}{2}} & 0 \\ 0 & R^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix},$$

Exogenous Inputs

$$\begin{bmatrix} w_d \\ w_n \end{bmatrix} = \begin{bmatrix} W^{\frac{1}{2}} & 0 \\ 0 & V^{\frac{1}{2}} \end{bmatrix} w$$

Generalized Plant

$$G = \left[\begin{array}{c|cc|c} A & W^{\frac{1}{2}} & 0 & B \\ \hline Q^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & R^{\frac{1}{2}} \\ \hline C & 0 & V^{\frac{1}{2}} & 0 \end{array}\right]$$



LQG (Linear Quadratic Gaussian)

$$J = E\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T z(t)^T z(t) dt\right\} = \|F_l(G,K)\|_2^2$$

H_2 Norm

$$E\left\{\begin{bmatrix} w_d(t) \\ w_n(t) \end{bmatrix} \begin{bmatrix} w_d(\tau)^T & w_n(\tau)^T \end{bmatrix}\right\} = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \delta(t-\tau)$$

$$z^T z$$
$$= \begin{bmatrix} x^T & u^T \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$
$$= x^T Q x + u^T R u$$

[Ex.] Spinning Satellite: Building Interconnection   sysic

Nominal Model

$$P(s) = \begin{bmatrix} \frac{s-100}{s^2+100} & \frac{10s+10}{s^2+100} \\ \frac{-10s-10}{s^2+100} & \frac{s-100}{s^2+100} \end{bmatrix}$$

Multiplicative (Output) Uncertainty

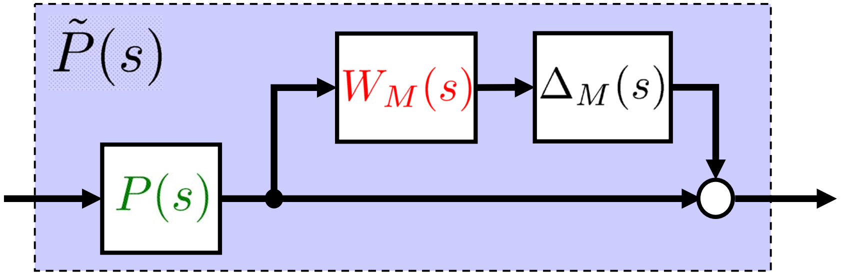
$$\Pi_0 = \{\tilde{P}(s) |\ \tilde{P}(s) = (I + \Delta_M(s) W_M(s)) P(s),\ \|\Delta_M\|_\infty \le 1\}$$

Uncertainty Weight

$$W_M(s) = w_M(s) I_2,$$

$$w_M(s) = \frac{0.045s + 0.4}{0.018s + 1}$$

$(\tau = 0.045,\ r_0 = 0.4,\ r_\infty = 2.5)$
$1/\tau = 22.2$

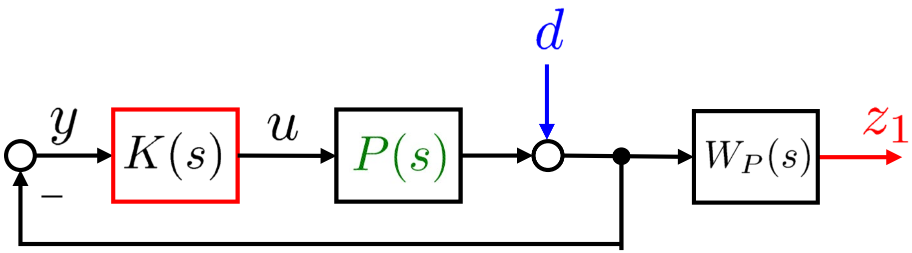


Performance Weight

$$W_P(s) = w_p(s) I_2\,,$$

$$w_p(s) = \frac{0.5s + 2}{s + 0.02}$$

$(\omega_b = 2,\ A = 0.01,\ M_S = 2)$



[Ex.] Spinning Satellite: Building Interconnection

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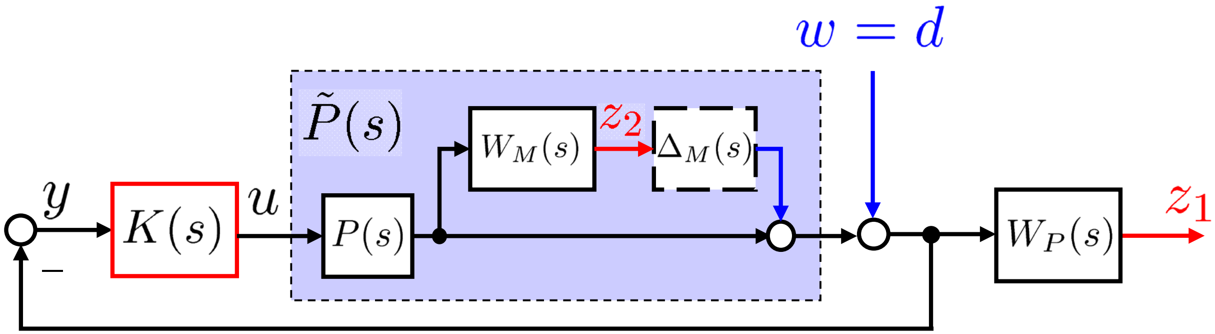
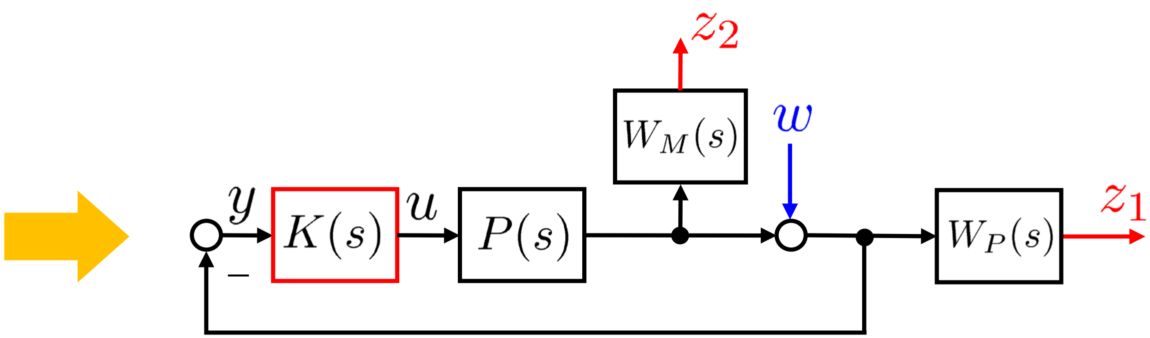
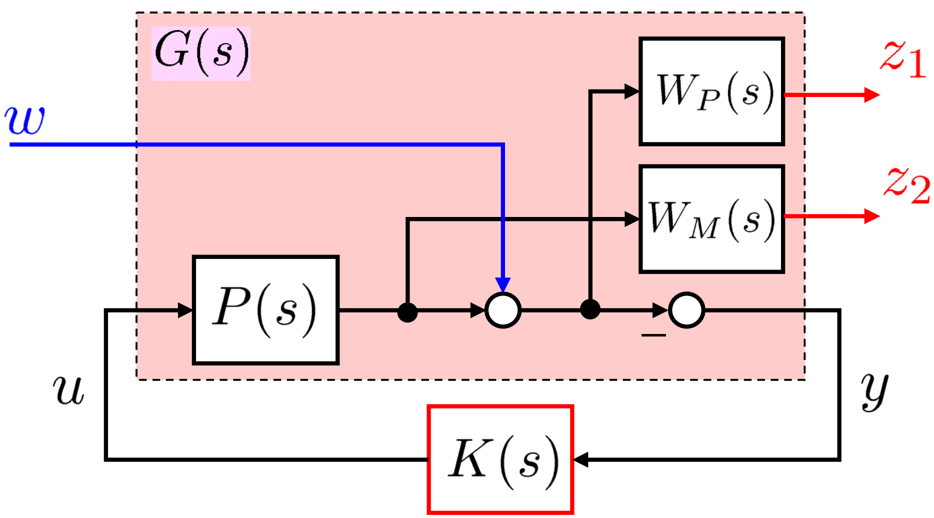




Interconnection (Mixed)

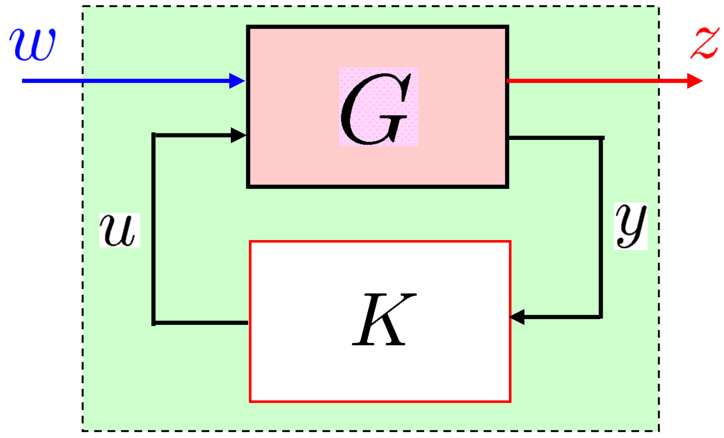


MATLAB Command

```
%Generalized Plant%
systemnames = ‘Pnom WP WM';
inputvar = '[w(2);u(2)]';
outputvar = '[WP;WM;-w-Pnom]';
input_to_Pnom= '[u]';
input_to_WP = '[w+Pnom]';
input_to_WM = ‘[Pnom]';
G = sysic;
```

Examples of H_∞ Control Problem [SP05, pp. 104-114]

- Sensitivity Minimization Problem
- Robust Stabilization Problem
- Mixed Sensitivity Problem
- LQG Type Control Problem
- Feedforward Problem
- Estimation Problem



Interconnection

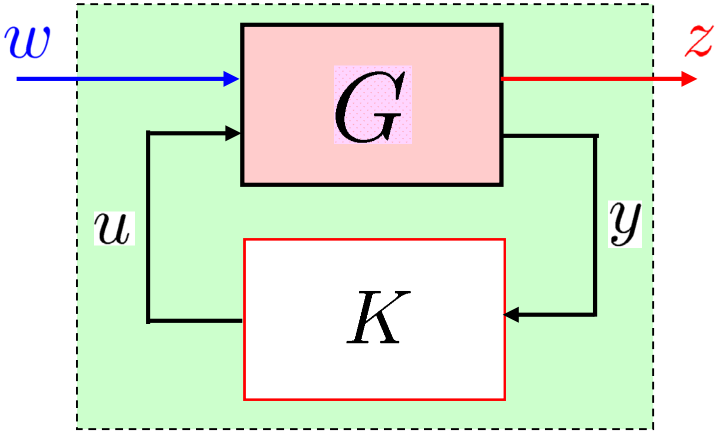
Nominal Plant Model P

Performance Weight W_P

Uncertainty Weight W_M

"If the Robust Control Toolbox of MATLAB complains, then it probably means that your control problem is not well formulated and you should think again"

H_∞ Control Problem [SP05, p. 357]



$$z = F_l(G, K)w$$

H_∞ Optimal Control Problem

Find all stabilizing controllers K which minimize

$$\|F_l(G, K)\|_\infty = \max_\omega \bar{\sigma}(F_l(G, K)(j\omega))$$

H_∞ Sub-optimal Control Problem

Given $\gamma > \gamma_{min}$, find all stabilizing controllers K such that

$$\|F_l(G, K)\|_\infty < \gamma$$

γ -iteration

The "1984" Approach (1984 ONR/Honeywell Workshop)

$$\begin{bmatrix} z \\ y \end{bmatrix} = G(s) \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

$$u = K(s)y$$

Generalized Plant

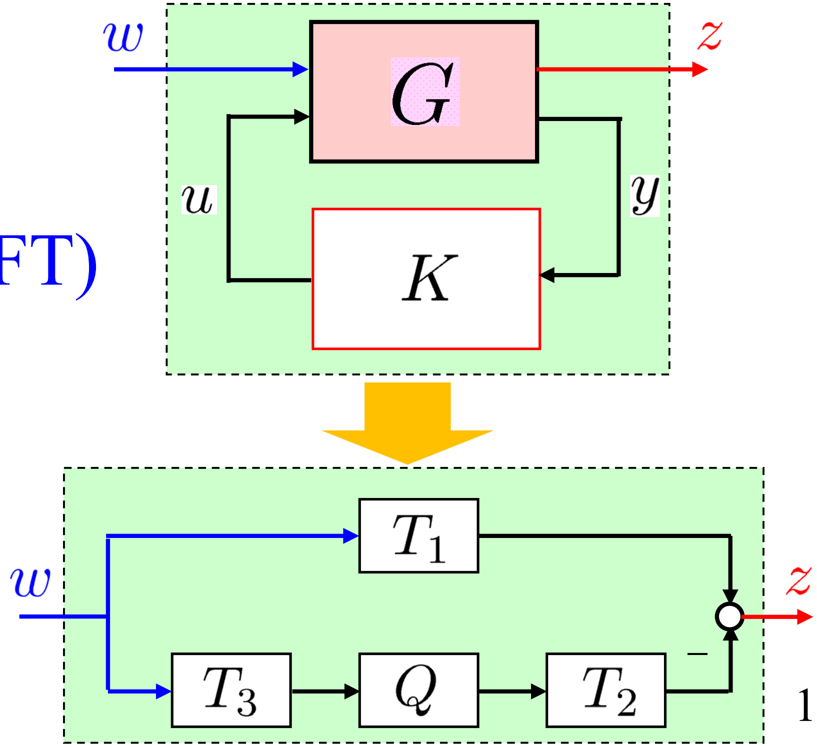
$$G(s) = M^{-1}N$$

All Stabilizing Controllers

$$K(s) = (Y - QN)^{-1}(X + QM)$$

$Q(s)$: Stable Transfer Function Matrix

B. A. Francis and J. C. Doyle, SIAM, 25-4, 1987



Closed-loop Transfer Function (LFT)

$$F_l(G, K) = T_1 - T_2 Q T_3$$

Model Matching Problem

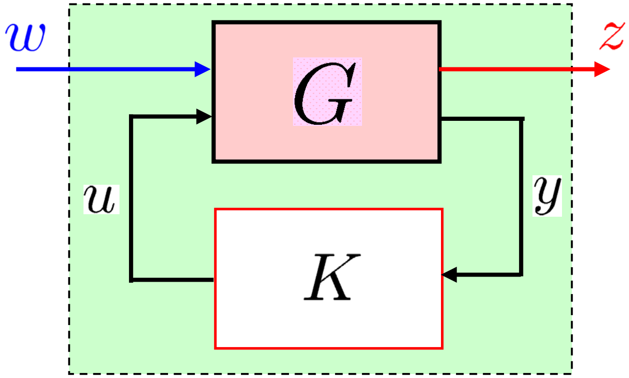
$$\|T_1(s) - T_2(s)Q(s)T_3(s)\|_\infty < \gamma$$

Affine in Q

State Space Approach [SP05, p. 357]

$$\begin{bmatrix} z \\ y \end{bmatrix} = G(s) \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

$$u = K(s)y$$



Generalized Plant

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned}$$

Closed-loop Transfer Function (LFT)

$$F_l(G, K) = G_{11} + G_{12} K (I - G_{22} K)^{-1} G_{21}$$

H_∞ Control Problem

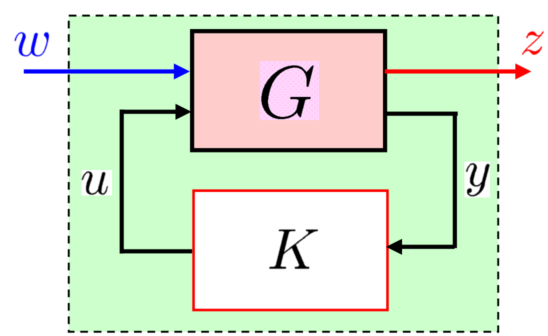
Given $\gamma > \gamma_{min}$, find all stabilizing controllers K such that

$$\|F_l(G, K)\|_\infty < \gamma$$

A Simplified H_∞ Control Problem [SP05, p. 353]

Generalized Plant

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \qquad \begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{12} u \\ y &= C_2 x + D_{21} w \end{aligned}$$



Assumptions

(A1) (A, B_2) is stabilizable and (C_2, A) is detectable

(A2) (A, B_1) is controllable and (C_1, A) is observable

[Full rank on the imaginary axis]

(A3) $D_{12}^T [C_1 \;\; D_{12}] = [0 \quad I]$ and $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$

$$\left(\begin{aligned} & z^T z = [\, x^T \;\; u^T \,]^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = x^T x + u^T u \\ & Q = C_1^T C_1 = I\,, \;\; R = D_{12}^T D_{12} = I \;, \;\; S = D_{12}^T C_1 = 0 \end{aligned} \right)$$

DGKF Solution [SP05, p. 357]

There exists a stabilizing controller K such that $\|F_l(G,K)\|_\infty < \gamma$ if and only if the following three conditions hold:

(i) There exists a solution $X_\infty \geq 0$ to

$$X_\infty A + A^T X_\infty + X_\infty(\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X_\infty + C_1^T C_1 = 0$$

such that $Re\ \lambda_i \left[A + (\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X_\infty\right] < 0,\ ^\forall i$

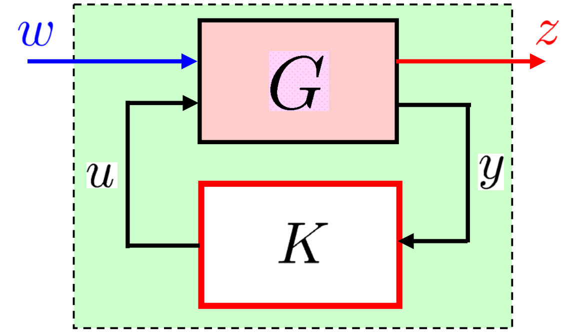
(ii) There exists a solution $Y_\infty \geq 0$ to

$$A Y_\infty + Y_\infty A^T + Y_\infty(\gamma^{-2} C_1^T C_1 - C_2^T C_2) Y_\infty + B_1 B_1^T = 0$$

such that $Re\ \lambda_i \left[A + Y_\infty(\gamma^{-2} C_1 C_1^T - C_2 C_2^T)\right] < 0,\ ^\forall i$

(iii) $\rho(X_\infty Y_\infty) < \gamma^2$

Central Controller [SP05, p. 358]

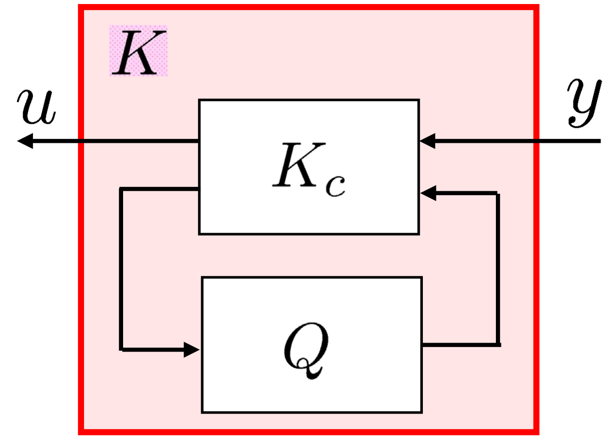


$$K_{sub}(s) = \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

$$\begin{cases} \hat{A}_\infty = A + \gamma^{-2} B_1 B_1^T X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2 \\ F_\infty = -B_2^T X_\infty \\ L_\infty = -Y_\infty C_2^T \\ Z_\infty = (I - \gamma^{-2} Y_\infty X_\infty)^{-1} \end{cases}$$

All H_∞ Controllers [SP05, p. 358]

$$K = F_l(K_c, Q)$$



$$K_c(s) = \left[\begin{array}{c|c:c} \hat{A}_\infty & -Z_\infty L_\infty & Z_\infty B_2 \\ \hline F_\infty & 0 & I \\ \hdashline -C_2 & I & 0 \end{array}\right]$$

$Q(s)$: Stable Proper Transfer Function Matrix such that $\|Q\|_\infty < \gamma$

Parameterization of H_∞ Control

DGKF

Doyle, Glover, Khargonekar, Francis, IEEE TAC, 34 - 8, 1989

(1988 ACC)

State-Space Solution to Standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Control Problems

J. C. Doyle

K. Glover

P. P. Khargonekar

B. A. Francis

Sketch of Proof (sufficiency) [Zhou98]

By using the following lemma, we propose one of $P_c \geq 0$ for feedback systems which consists of $P(s)$ and $K(s)$.

Bounded Real Lemma

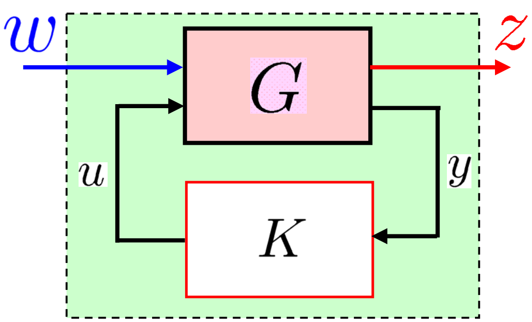
For $\gamma > 0$, $F_l = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right]$, the following two conditions are equivalent.

(i) $\|F_l\|_\infty < \gamma$

(ii) There exists a $P_c > 0$ such that

$$P_c A_c + A_c^T P_c + \gamma^{-2} P_c B_c B_c^T P_c + C_c^T C_c = 0$$

and $A_c + \gamma^{-2} B_c B_c^T P_c$ has no eigenvalues on the imaginary axis.



$z = F_l(G, K) w$

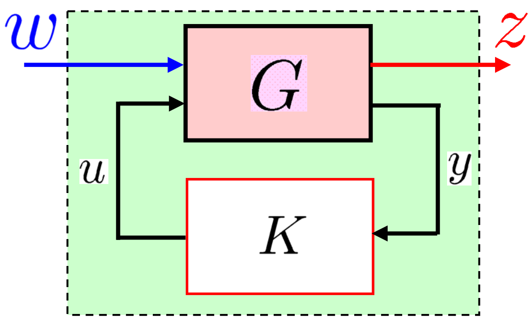
[Zhou98] K. Zhou with J.C. Doyle, *Essentials of Robust Control*, Prentice Hall, 1998.

Case 1: state $x_c = \begin{bmatrix} x^T & \hat{x}^T \end{bmatrix}^T$

Consider the feedback loop with the state $x_c = \begin{bmatrix} x^T & \hat{x}^T \end{bmatrix}^T$.

Generalized Plant $\begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{12} u \\ y = C_2 x + D_{21} w \end{cases}$

Central Controller $\begin{cases} \dot{\hat{x}} = \hat{A}_\infty \hat{x} + (-Z_\infty L_\infty) y \\ u = F_\infty \hat{x} \end{cases}$



$z = F_l(G, K) w$

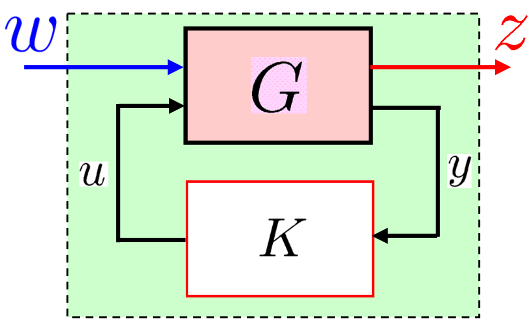
Closed-loop Transfer Function (LFT) from w to z

$$F_l = \left[\begin{array}{cc|c} A & B_2 F_\infty & B_1 \\ -Z_\infty L_\infty C_2 & \hat{A}_\infty & -Z_\infty L_\infty D_{21} \\ \hline C_1 & D_{12} F_\infty & 0 \end{array}\right] = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right]$$

$$\left(\begin{array}{l} \dfrac{d}{dt}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} A & B_2 F_\infty \\ -Z_\infty L_\infty C_2 & \hat{A}_\infty \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B_1 \\ -Z_\infty L_\infty D_{21} \end{bmatrix} w \\ z = \begin{bmatrix} C_1 & D_{12} F_\infty \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + 0 \cdot w \end{array} \right)$$

Define P_c by

$$P_c = \begin{bmatrix} \gamma^2 Y_\infty^{-1} & -\gamma^2 Y_\infty^{-1} Z_\infty^{-1} \\ -\gamma^2 (Z_\infty^T)^{-1} Y_\infty^{-1} & \gamma^2 Y_\infty^{-1} Z_\infty^{-1} \end{bmatrix}$$



Then, we have

$$P_c > 0$$

$$P_c A_c + A_c^T P_c + \gamma^{-2} P_c B_c B_c^T P_c + C_c^T C_c = 0$$

and

positive semi-definite

$$A_c + \gamma^{-2} B_c B_c^T P_c = \begin{bmatrix} A + B_1 B_1^T Y_\infty^{-1} & B_2 F_\infty - B_1 B_1^T Y_\infty^{-1} Z_\infty^{-1} \\ 0 & A + \gamma^{-2} B_1 B_1^T X_\infty + B_2 F_\infty \end{bmatrix}$$

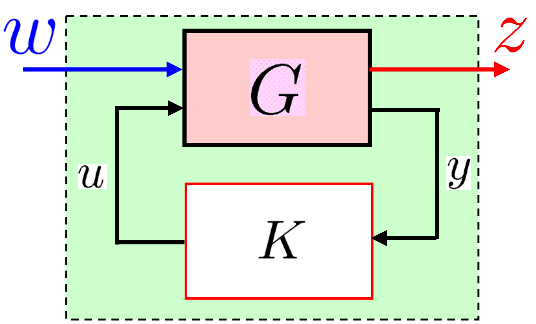
positive definite

does not have eigenvalues on the imaginary axis.

Hence, from Bounded Real Lemma, there exists a stabilizing controller K such that $\|F_l(G,K)\|_\infty < \gamma$. □

M. Sampei, T. Mita and M. Nakamichi, "An Algebraic Approach to H_∞ Output Feedback Control Problems," Systems and Control Letters, Vol. 14, pp. 13-24, 1990.

Case 2: state $x_c = \left[x^T \ (x-\hat{x})^T\right]^T$



Consider the feedback loop with the state

$$x_c = \left[x^T \ (x-\hat{x})^T\right]^T$$

$$z = F_l(G,K)w$$

Closed-loop Transfer Function (LFT) from w to z

$$F_l = \left[\begin{array}{cc|c} A+B_2F_\infty & -B_2F_\infty & B_1 \\ A+Z_\infty L_\infty C_2 + B_2F_\infty - \hat{A}_\infty & \hat{A}_\infty - B_2F_\infty & B_1 + Z_\infty L_\infty D_{21} \\ \hline C_1 + D_{12}F_\infty & -D_{12}F_\infty & 0 \end{array}\right] = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right]$$

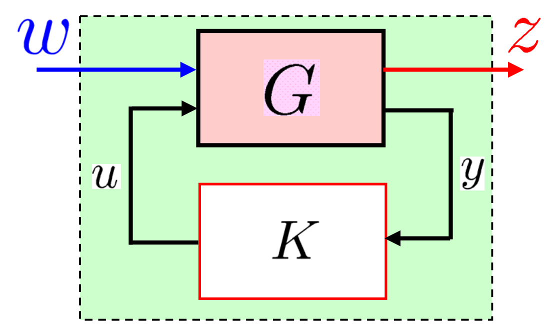
Set P_c as follows

$$P_c = \begin{bmatrix} X_\infty & 0 \\ 0 & \gamma^2 T^{-1} \end{bmatrix}, \ T = \left(I - \gamma^{-2} Y_\infty X_\infty\right)^{-1} Y_\infty$$

The remaining part is the same as the previous one. □

K. Uchida and M. Fujita, "On the central controller: Characterizations via differential games and LEQG control problems," *Systems and Control Letter*, vol. 13, no. 1, pp. 9-13, 1989.

Case 3: state $x_c = \left[\hat{x}^T \ (x-\hat{x})^T\right]^T$



Consider the feedback loop with the state

$$x_c = \left[\hat{x}^T \ (x-\hat{x})^T\right]^T$$

Closed-loop Transfer Function (LFT) from w to z $\quad z = F_l(G,K)w$

$$F_l = \left[\begin{array}{cc|c} A + B_2 F_\infty Z_\infty + \gamma^{-2} P_c C_1' C_1 & -L_\infty C_2 & -L_\infty D_{21} \\ -\gamma^{-2} P_c C_1' C_1 & L_\infty C_2 + A & B_1 + L_\infty D_{21} \\ \hline C_1 + D_{12} F_\infty Z_\infty & C_1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right]$$

Set P_c as follows

$$P_c = \begin{bmatrix} S_\infty & 0 \\ 0 & \gamma^2 Y_\infty^{-1} \end{bmatrix}, \ S = X_\infty \left(I - \gamma^{-2} Y_\infty X_\infty\right)^{-1}$$

The remaining part is the same as the previous one. □

K. Uchida and M. Fujita, "Finite Horizon H_∞ Control Problems with Terminal Penalties," *IEEE Trans. on Automatic Control*, vol. 37, no. 11, pp. 1762-1767, 1992.

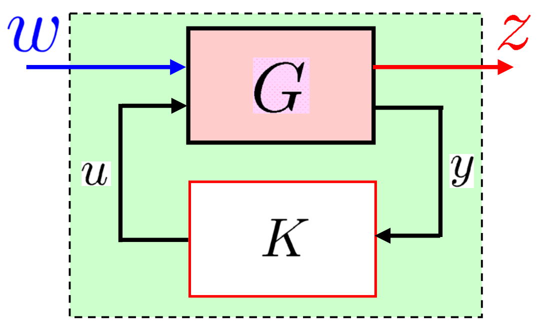
Structure of H_∞ Central Controller [SP05, p. 358]

$$u = F_\infty \hat{x}\,,\;\; F_\infty = -B_2^T X_\infty$$: State Feedback

$$\hat{w}_{worst} = \gamma^{-2} B_1^T X_\infty \hat{x}$$: Worst Disturbance Estimation

$$\dot{\hat{x}} = A\hat{x} + B_1 \gamma^{-2} B_1^T X_\infty \hat{x} + B_2 u + Z_\infty L_\infty (C_2 \hat{x} - y)$$

: Worst State Estimation



$$z = F_l(G, K) w$$

Minimum Entropy Controller

$$\|T\|_\infty < \gamma$$

Entropy

$$I(T,\gamma) = -\frac{\gamma^2}{2\pi}\int_{-\infty}^{\infty} \ln|\det(I - \gamma^{-2} T^*(j\omega) T(j\omega))| d\omega$$

$$\Rightarrow \min_K I(F_l, \gamma)$$

From LQG Control to H_∞ Control

LQG Controller (H_2 Controller)

$\gamma \to \infty$

State Feedback

$$u = F_\infty \hat{x} \qquad F_\infty = -B^T X_\infty$$

State Estimation

$$\dot{\hat{x}} = A\hat{x} + Bu + L_\infty(C\hat{x} - y)$$

$$L_\infty = -Y_\infty C^T$$

Riccati Equations: $X_\infty \geq 0, Y_\infty \geq 0$

$$X_\infty A + A^T X_\infty - X_\infty B B^T X_\infty + C^T C = 0$$

$$A Y_\infty + Y_\infty A^T - Y_\infty C^T C Y_\infty + B B^T = 0$$

H_∞ Central Controller

State Feedback

$$u = F_\infty \hat{x} \qquad F_\infty = -B_2^T X_\infty$$

Worst Disturbance Estimation

$$\hat{w}_{worst} = \gamma^{-2} B_1^T X_\infty \hat{x}$$

Worst State Estimation

$$\dot{\hat{x}} = A\hat{x} + B_1 \hat{w}_{worst} + B_2 u + Z_\infty L_\infty (C_2 \hat{x} - y)$$

$$L_\infty = -Y_\infty C_2^T$$

$$Z_\infty = (I - \gamma^{-2} Y_\infty X_\infty)^{-1}$$

Riccati Equations: $X_\infty \geq 0, Y_\infty \geq 0$

$$X_\infty A + A^T X_\infty + X_\infty(\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X_\infty + C_1^T C_1 = 0$$

$$A Y_\infty + Y_\infty A^T + Y_\infty(\gamma^{-2} C_1^T C_1 - C_2^T C_2) Y_\infty + B_1 B_1^T = 0$$

LQ Theory

Linear System

$$\dot{x} = Ax + Bu$$

Cost Function

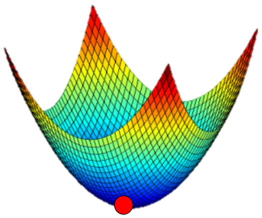
$$J = \int_{t_0}^{\infty} \left(x^T Q x + u^T R u \right) dt$$

$$Q \geq 0, \ \ R > 0$$

u minimizes cost function J

For Certain u^o

$$J(u^o) \leq J(u)$$



Game Theory

Linear System

$$\dot{x} = Ax + Bu + Dw$$

Cost Function

$$J = \int_{t_0}^{\infty} \left(x^T Q x + u^T R u - w^T N w \right) dt$$

$$Q \geq 0, \ \ R > 0, \ \ N > 0$$

u minimizes cost function J

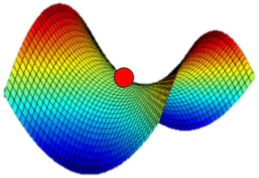
w maximizes cost function J

For Certain (u^o, w^o)

$$J(u^o, w) \leq J(u^o, w^o) \leq J(u, w^o)$$

Output feedback

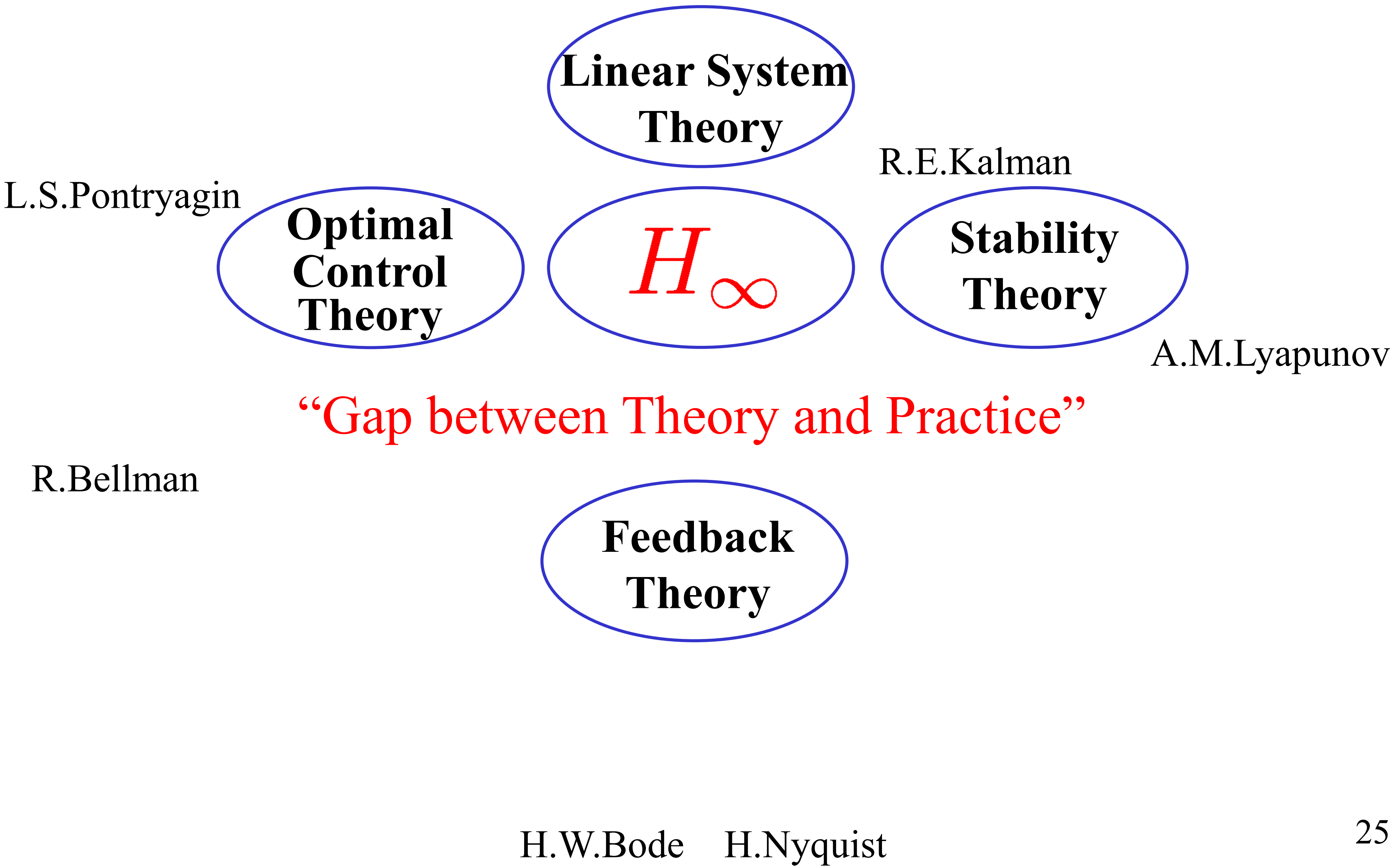
: Uchida and Fujita, 1989



LEQG Control

"E": Exponential

From LQG Control to H_∞ Control



Completion of Modern Control Theory

A stabilizing controller
State feedback/Observer

An optimal controller
LQG(=LQR+Kalman Filter)

All stabilizing controllers
Q (Youla) Parametrization

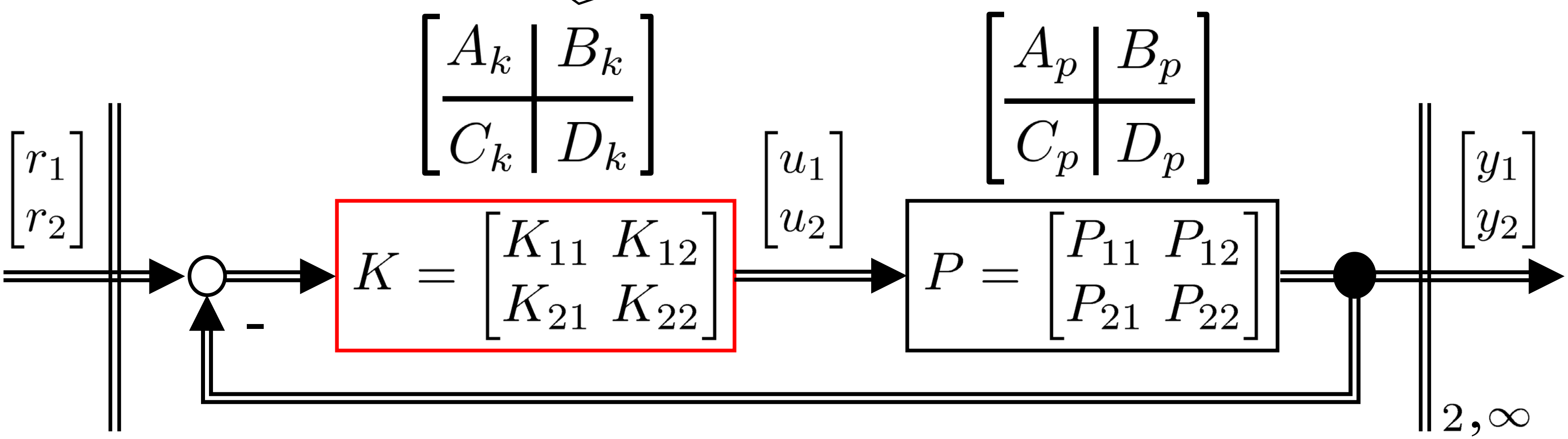
All optimal controllers
H_∞ controller
($\gamma \to \infty$: H_2=LQG)

Transfer Function
Pole/Zero

Structure
Controllability,
Observability

State Space Form
(Data Structure)

State x



Robust and Optimal Control

1940's –1950s→

1960's –1970s→

Tough and Strong

Smart and Intelligent

1980's –1990s→

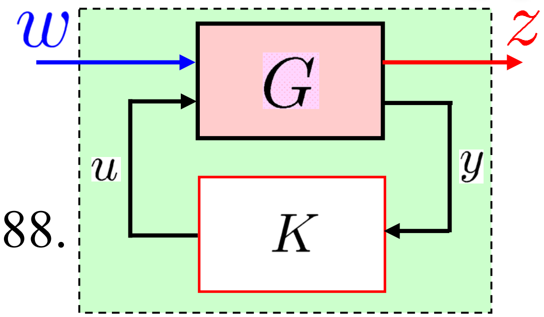
Tough and Smart

[ZDG96] K. Zhou, J. Doyle and K. Glover, *Robust and Optimal Control*, Prentice Hall, 1996

General H_∞ Solutions

"State-space Formulate for All Stabilizing Controllers that Satisfy an H_∞ Norm Bound and Relations to Risk Sensitivity"

K. Glover and J.C. Doyle, Systems and Control Letters, 11, 1988.



hinfsyn (h2syn 5)

$$\|F_l(G,K)\|_\infty < \gamma$$

[k, cl, gam, info] = hinfsyn (p, nmeas, ncon, key1, value1, key2, value2, …)

input argument

- p — generalized plant
- nmeas — number of measurement outputs
- ncon — number of control inputs

output argument

- k — LTI controller
- cl — closed loop system which consists of K and G
- gam — H_∞ norm of closed loop system
- info — information of output results

Key setting

- Gmax — upper limit of Gam
- Gmin — lower limit of Gam
- Tolgam — relative error of Gam
- So — frequency at which entropy is assessed
- Method (8)
  - Ric : Ricatti solution
  - Lmi : LMI solution
  - Maxe : max entropy solution
- Display
  - Off : not show setting process
  - On : show setting process

Robust Control Toolbox

Robust Control Toolbox (1988) μ - Analysis and Synthesis Toolbox (1993) LMI Control Toolbox (1995)

Robust Control Toolbox ver. 3 (2005~) R2019a

(Eds.) Gary Balas Andy Packard Michael Safonov

After DGKF

[DP05] G.E. Dullerud and F. Paganini,
A Course in Robust Control Theory:
A Convex Approach,
Text in Applied Mathematics, Springer, 2005.

- Linear Matrix Inequality (LMI)
- Linear Parameter Varying (LPV) Systems
- Integral Quadratic Constraints (IQC)
- Sum of Squares (SOS)

System Level Synthesis

1940's –1950's→

Tough and Strong

1960's –1970's→

Smart and Intelligent

1980's –1990's→

Tough and Smart

2000's – 2010's →

Tough, Smart and Elegant

System Level Synthesis (SLS)

K. Zhou, J. Doyle and K. Glover, *Robust and Optimal Control*, Prentice Hall, 1996

J. Anderson, J. Doyle, S. Low, and N. Matni, "System Level Synthesis," Annual Reviews in Control, to appear, 2019

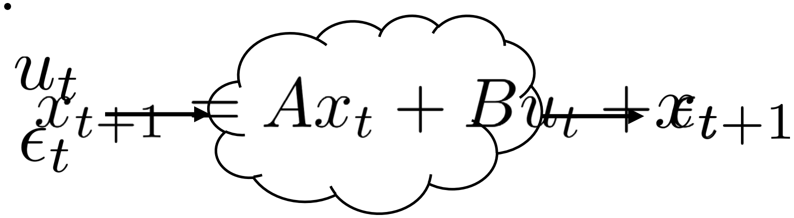
A.I. = Actionable Intelligence

Model-based / free LQR

minimize $\mathrm{E}\left[\sum_{t=1}^{T} x_t^T Q x_t + u_t^T R u_t\right]$

s.t.

$$x_{t+1} = Ax_t + Bu_t + \epsilon_t$$



$$u_t = Kx_t$$

Reinforcement Learning

P.P. Khargonekar, and M.A. Dahleh,
Advancing systems and control research in the era of ML and AI,
Annual Reviews in Control, Vol. 45, pp. 1-4, 2019

5. H_∞ Control

- ✓ 5.1 General Control Problem Formulation

[SP05, Sec. 3.8]

- ✓ 5.2 H_∞ Control Problem and DGKF Solutions

[SP05, Sec. 9.3]

- ✓ 5.3 Structure of H_∞ Controllers

Reference:

[SP05] S. Skogestad and I. Postlethwaite, ***Multivariable Feedback Control; Analysis and Design,*** Second Edition, Wiley, 2005.

6. Design Example

6.1 Spinning Satellite: H_∞ Control [SP05, Sec. 3.7]

6.2 2nd Report

Reference:

[SP05] S. Skogestad and I. Postlethwaite,
Multivariable Feedback Control; Analysis and Design,
Second Edition, Wiley, 2005.

An Example of Generalized Plant

F-18 High Alpha Research Vehicle

Flight Control Design Using Robust Dynamic Inversion and Time-scale Separation

J. Reiner, G. J. Balas and W. L. Garrard, Automatica, 32 - 11, 1996

Model Matching Problem in SISO Systems

$P(s)$: Stable

All Stabilizing Controllers

B.A. Francis, Springer-Verlag, 1987

$$K(s) = \frac{Q(s)}{1 - P(s)Q(s)}\ , \quad Q(s) : \text{Proper Stable}$$

Decision of Parameter $Q(s)$

Nominal Performance $\|w_P(s)S(s)\|_\infty < 1 \ \ (S(s) = 1 - P(s)Q(s))$

$$\|w_P(s)S(s)\|_\infty = \|w_P(s) - w_P(s)P(s)Q(s))\|_\infty < 1$$

$$\Rightarrow \quad T_1(s) = w_P(s),\ T_2(s) = w_P(s)P(s),\ T_3(s) = 1,\ \gamma = 1$$

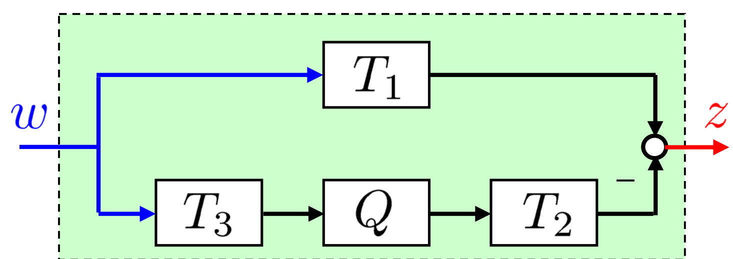
Robust Stabilization $\|w_M(s)T(s)\|_\infty < 1 \ \ (T(s) = P(s)Q(s))$

$$\|w_M(s)T(s)\|_\infty = \|w_M(s)P(s)Q(s)\|_\infty < 1$$

$$\Rightarrow \quad T_1(s) = 0,\ T_2(s) = w_M(s)P(s),\ T_3(s) = 1,\ \gamma = 1$$

Model Matching Problem

$$\|T_1(s) - T_2(s)Q(s)T_3(s)\|_\infty < \gamma$$



Assumptions of H_∞ Control Problem for simplicity

[SP05, p. 354]

The following assumptions are typically made in H_2 and H_∞ problems:

(A1) (A, B_2) is stabilizable and (C_2, A) is detectable

[$\because$ Requirement for the existence of stabilizing controllers K]

(A2) D_{12} and D_{21} have full rank

[$\because$ Sufficient to ensure the controllers are proper and hence realizable]

(A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω

(A4) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω

[$\because$ To ensure that the optimal controller does not try to cancel poles or zeros on the imaginary axis which would result in closed-loop instability]

(A5) $D_{11} = 0$ and $D_{22} = 0$

[$\because$ Conventionality in H_2 control. $D_{11} = 0$ makes G_{11} strictly proper. $D_{22} = 0$ makes G_{22} strictly proper and simplifies the formulas in the algorithms.]

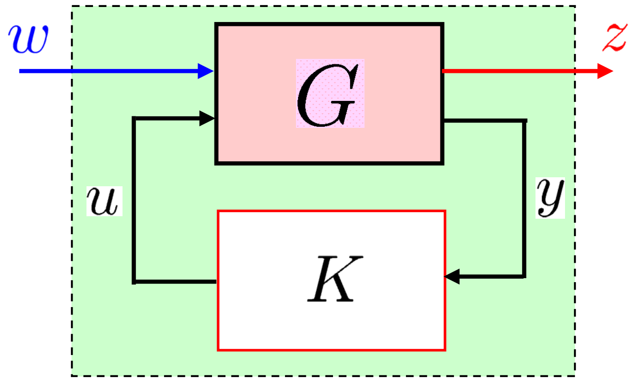
[$\because$ Neither of them are required in H_∞ control. For significant simplicity.]

Assumptions of H_∞ Control Problem for simplicity

It is also sometimes assumed that D_{12} and D_{21} are given by

(A6) $D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$ and $D_{21} = \begin{bmatrix} 0 & I \end{bmatrix}$

[$\because$ This can achieved, without loss of generality, by a scaling of u and y and a unitary transformation of w and z .]



In addition, for simplicity of exposition, the following assumptions are sometimes made

(A7) $D_{12}^T C_1 = 0$ and $B_1 D_{21}^T = 0$

(A8) (A, B_1) is controllable and (C_1, A) is observable

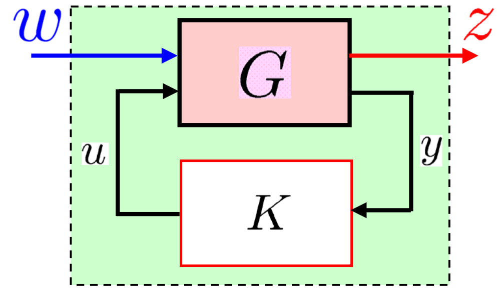
[If (A7) holds, then (A3) and (A4) may be replaced by (A8)]

[From (A2), (A6) and (A7), the following equations hold]

$$D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix} \text{ and } \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

H_2 Control Problem [SP05, pp. 344-351, 355-357]

H_2 Optimal Control Problem



$$z = F_l(G, K)w$$

Find a stabilizing controller K which minimize

$$\|F_l(G,K)\|_2 = \sqrt{\frac{1}{2\pi}\int_{\infty}^{\infty} \mathrm{tr}[F_l(j\omega)F_l(j\omega)^H]d\omega}$$

$$\dot{x} = Ax + Bu + w_d$$

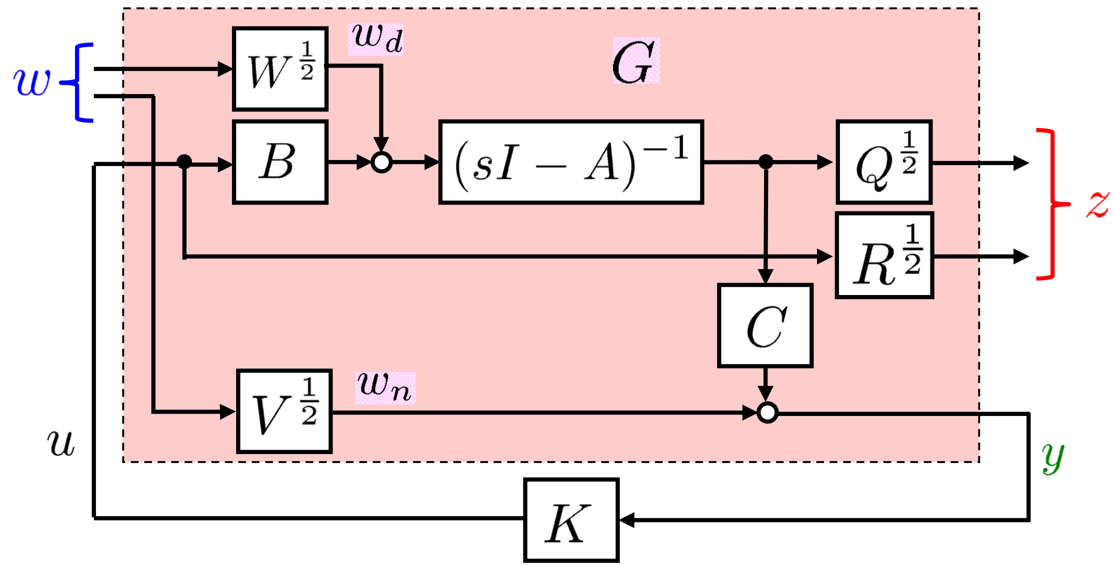
$$y = Cx + w_n$$

Regulated Outputs

$$z = \begin{bmatrix} Q^{\frac{1}{2}} & 0 \\ 0 & R^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

Exogenous Inputs

$$\begin{bmatrix} w_d \\ w_n \end{bmatrix} = \begin{bmatrix} W^{\frac{1}{2}} & 0 \\ 0 & V^{\frac{1}{2}} \end{bmatrix} w$$



h2syn

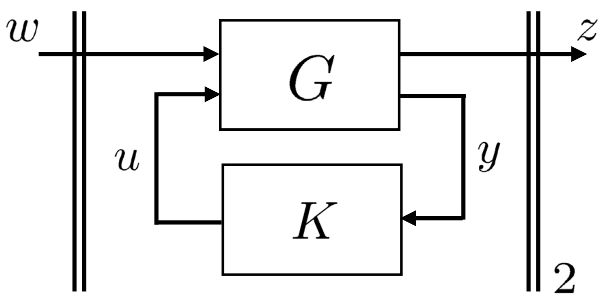
[k, cl, gam, info] = h2syn (p, nmeas, ncon)

H_∞ Control Solutions

H_∞ synthesis

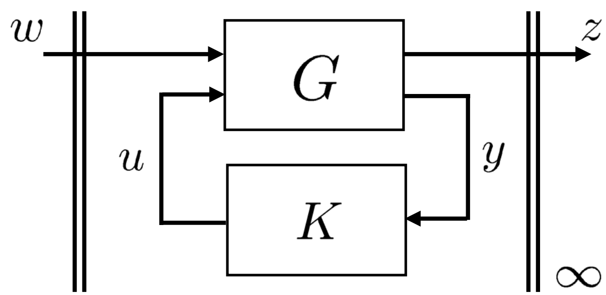
h2syn

H_2 controller synthesis



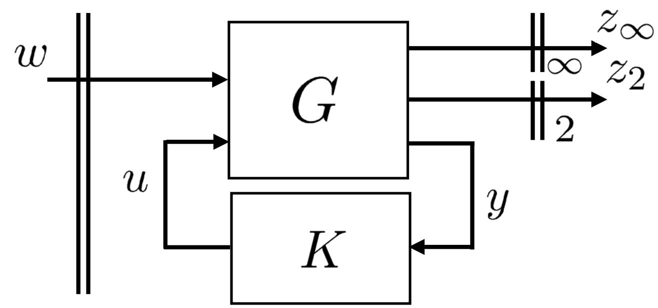
hinfsyn

H_∞ controller synthesis



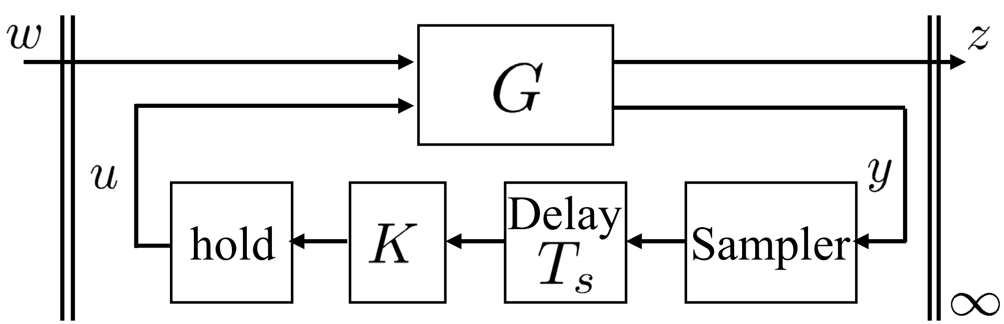
h2hinfsyn

Mixed H_2/H_∞ controller synthesis



sdhinfsyn

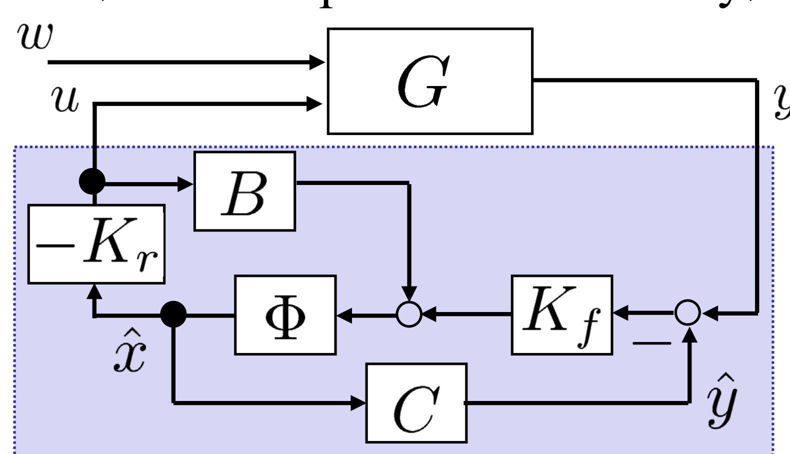
Sample-data H_∞ controller synthesis



ltrsyn

LQG LTR controller synthesis

(LTR: Loop-transfer Recovery)



H_∞ Control Solutions

Loop-shaping synthesis

loopsyn

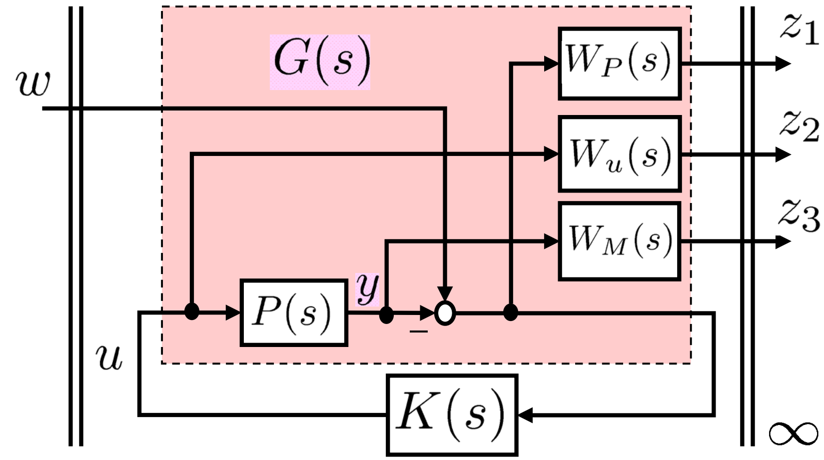
H_∞ loop shaping controller synthesis

ncfsyn

H_∞ irreducible decomposition controller synthesis (using Glover-McFarlane Method)

mixsyn

H_∞ mixed sensitivity controller synthesis



μ synthesis

dksyn

Robust controller design using μ -synthesis

LMI Solutions [SP05, Sec. 12]

Linear Matrix Inequality (LMI)

$$F(x) := F_0 + x_1 F_1 + x_2 F_2 + \cdots + x_m F_m > 0$$

$F_i (i = 0, 1, \cdots, m)$: Constant symmetric real matrices

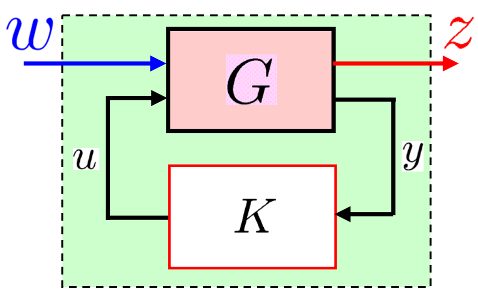
Riccati Inequality

$$A^T X + XA + (XB + S)R^{-1}(B^T X + S^T) + Q < 0$$

LMI
$$\begin{bmatrix} -A^T X - XA - Q & XB + S \\ B^T X + S^T & R \end{bmatrix} > 0$$

H_∞ Control Problem

$$\begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{11} w + D_{12} u \end{cases}$$



$u = Kx$: State Feedback

$z = F_l(G, K)w$

$\|F_l(G,K)\|_\infty < \gamma \Leftrightarrow$ There exist $\mathcal{X} = \mathcal{X}^T > 0, \mathcal{M} = K\mathcal{X}$ such that

$$\begin{bmatrix} A\mathcal{X} + B_2\mathcal{M} + (\mathcal{X}A + B_2\mathcal{M})^T + B_1 B_1^T & (C_1\mathcal{X} + D_{12}\mathcal{M})^T + B_1 D_{11}^T \\ C_1\mathcal{X} + D_{12}\mathcal{M} + D_{11}B_1^T & D_{11}D_{11}^T - \gamma^2 I \end{bmatrix} < 0$$