Robust Control

Spring, 2019

Instructor: Prof. Masayuki Fujita (S5-303B)

7th class

Tue., 28th May, 2019, 10:45～12:15,

S423 Lecture Room

7. Robust Performance

7.1 Robust Performance [SP05, Sec. 7.6, 8.3, 8.4, 8.10]

7.2 Structured Singular Value μ

[SP05, Sec. 8.5, 8.6, 8.8, 8.11]

7.3 μ-Analysis and Synthesis

[SP05, Sec. 7.6, 8.9, 8.10, 8.11]

Reference:

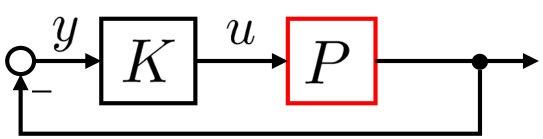
[SP05] S. Skogestad and I. Postlethwaite, ***Multivariable Feedback Control; Analysis and Design,*** Second Edition, Wiley, 2005.

Robust Performance in MIMO Systems [SP05, pp. 300, 316-320]

P : Nominal Plant Model

$\tilde{P} \in \Pi$: A Set of Plant Models

NS: Nominal Stability

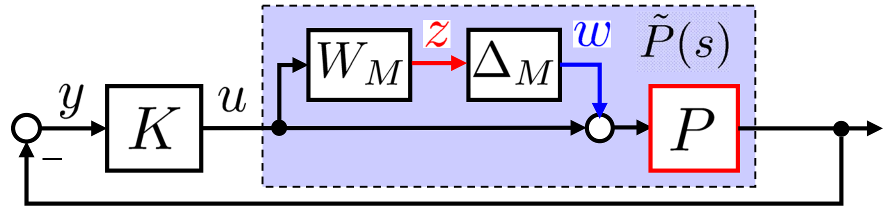


Gang of Four: Stable

(S, T, PS, KS)

Youla Parameterization

RS: Robust Stability

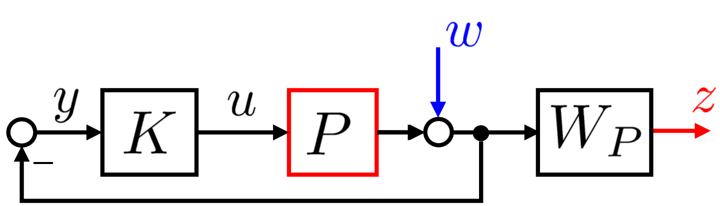


$$\|W_M T\|_\infty < 1$$

Δ_M : Unstructured Uncertainty

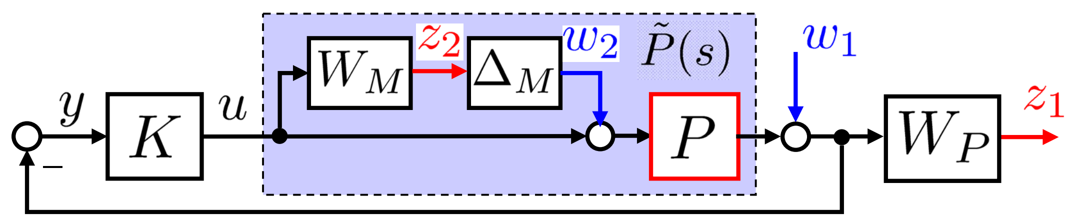
$$\|\Delta_M\|_\infty \leq 1$$

NP: Nominal Performance



$$\|W_P S\|_\infty < 1$$

RP: Robust Performance



$$\|W_P \tilde{S}\|_\infty < 1 \quad \forall \tilde{P} \in \Pi$$

$$\tilde{S} = (I + \tilde{P}K)^{-1}$$

Stability, Performance $\Rightarrow$ Robustness

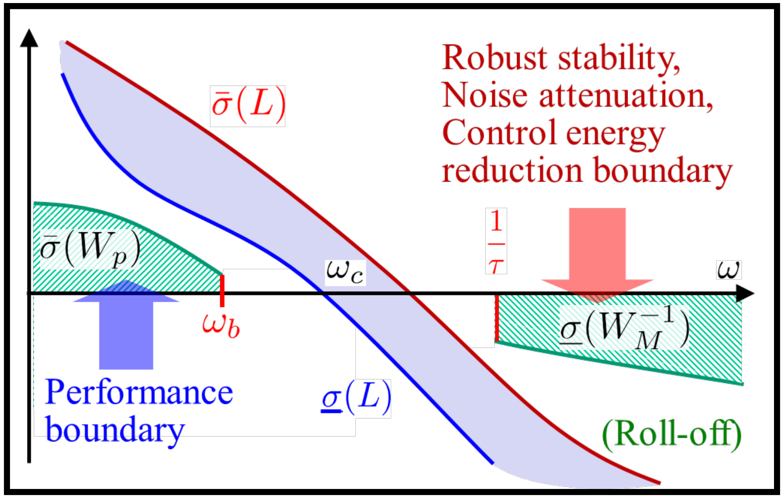
Beyond Multivariable Loop Shaping

G. Stein and J. C. Doyle,

"Beyond singular values and loop shapes,"

AIAA Journal of Guidance, Control and Dynamics, **14**-1, 5-16, 1991

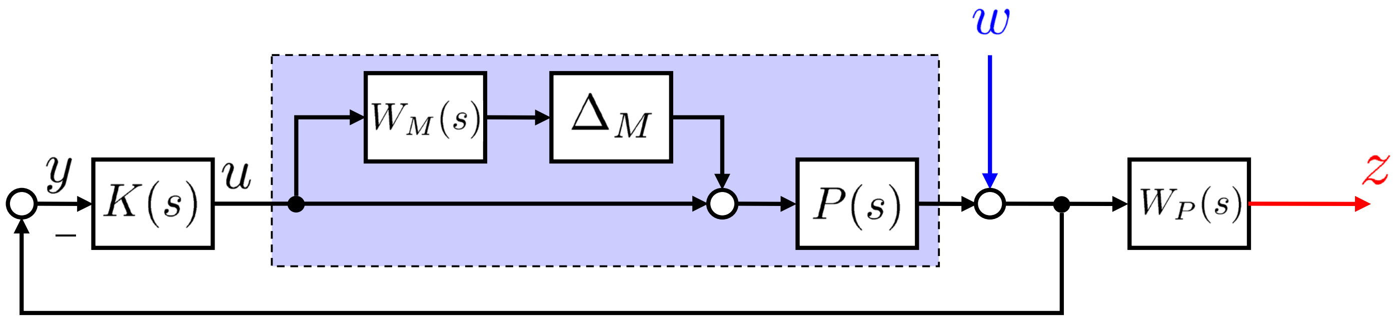
Gunter Stein   John C. Doyle

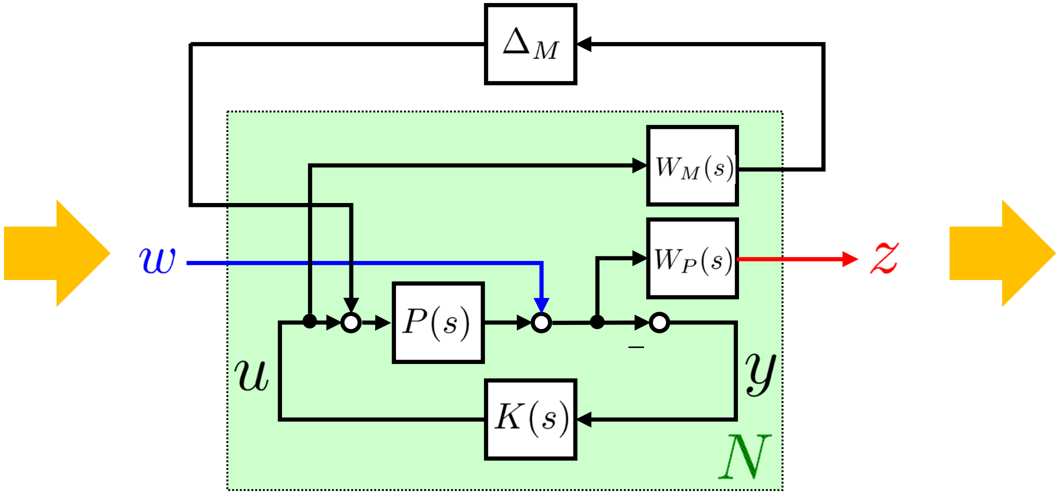


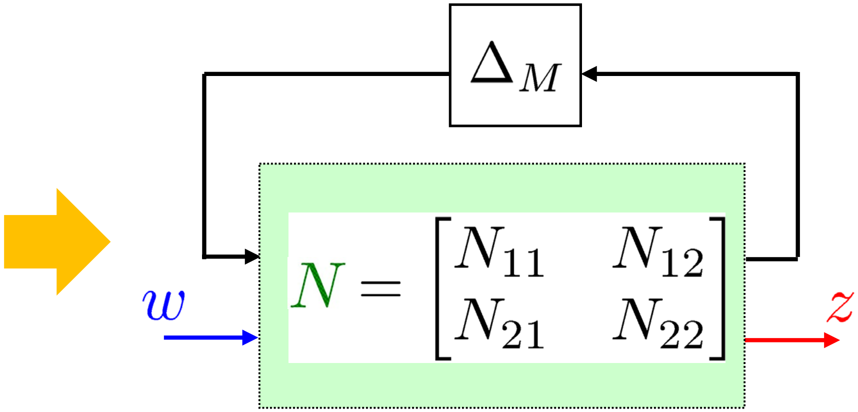
Multivariable Loop Shaping via Singular Values σ

Structured Singular Value μ (SSV)

Framework of Robust Performance Problem [SP05, p. 298]





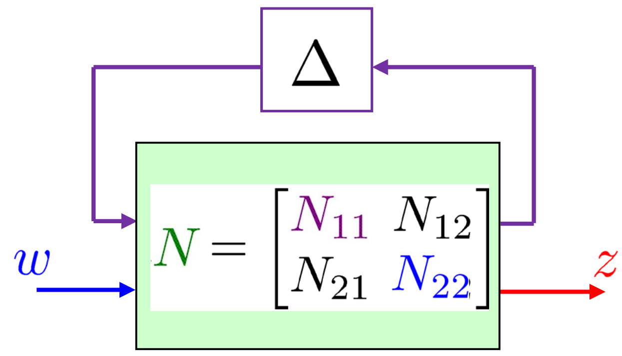


$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} -W_M KP(I+KP)^{-1} & -W_M K(I+PK)^{-1} \\ W_P(I+PK)^{-1}P & W_P(I+PK)^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} -W_M T_I & -W_M K S_o \\ W_P S_o P & W_P S_o \end{bmatrix}$$

NS, RS, NP, RP [SP05, p. 300]

Closed-loop Transfer Function (LFT)



$$z = F_u(N, \Delta)w$$

$$F_u(N, \Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}$$

NS: Nominal Stability

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} -W_M T_I & -W_M K S_o \\ W_P S_o P & W_P S_o \end{bmatrix}$$: Internally Stable

Gang of Four

NP: Nominal Performance

NS and $\|N_{22}\|_\infty = \|W_P S_o\|_\infty < 1 \quad (\mu(N_{22}(j\omega)) < 1, \ ^\forall\omega)$

RS: Robust Stability

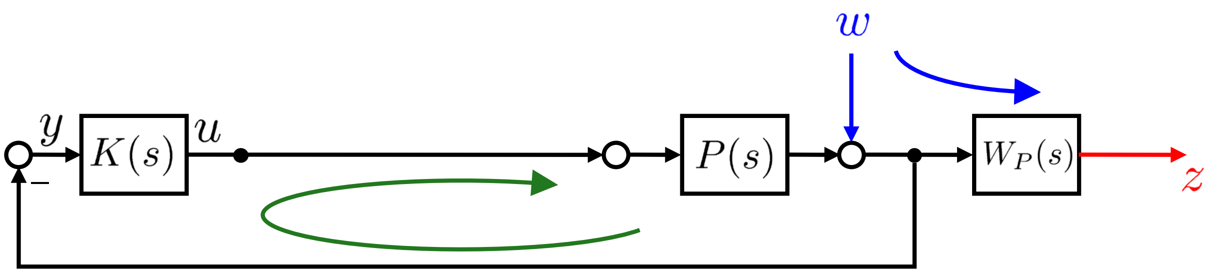
$$\|N_{11}\|_\infty = \|W_M T_I\|_\infty < 1 \quad (\mu(N_{11}(j\omega)) < 1, \ ^\forall\omega)$$

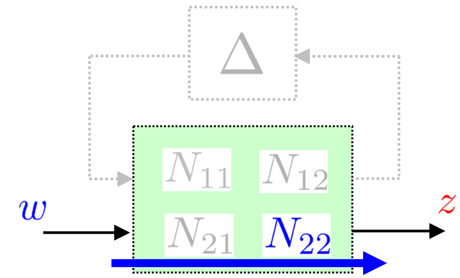
RP: Robust Performance

RS and $\|F_u(N, \Delta)\|_\infty < 1 \ ^\forall\Delta, \|\Delta\|_\infty \le 1$

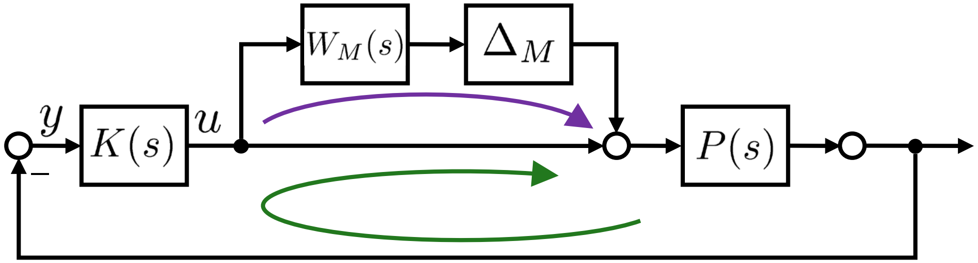
NS, RS, NP, RP (Cont'd)

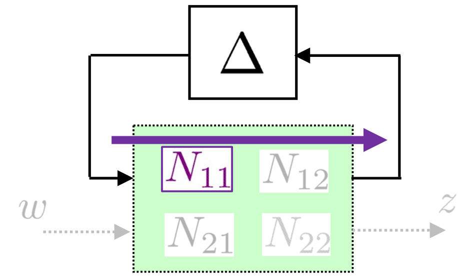
NP: Nominal Performance $\mu(N_{22}(j\omega)) < 1, \; ^\forall\omega$





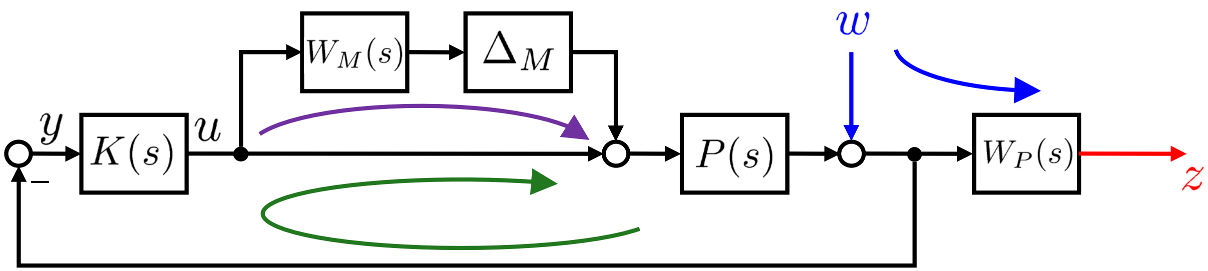
RS: Robust Stability $\mu(N_{11}(j\omega)) < 1, \; ^\forall\omega$

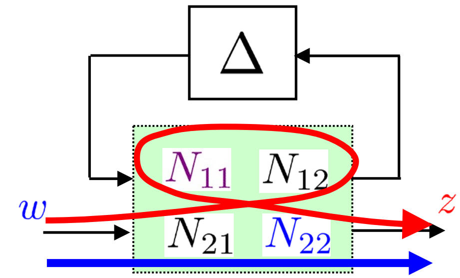




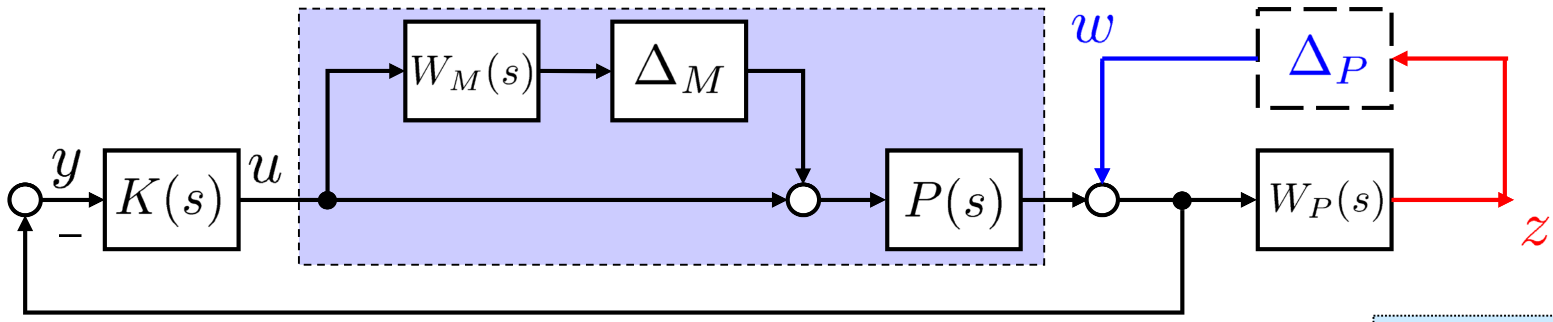
RP: Robust Performance $\|F_u(N,\Delta)\|_\infty < 1 \; ^\forall\Delta, \|\Delta\|_\infty \le 1$

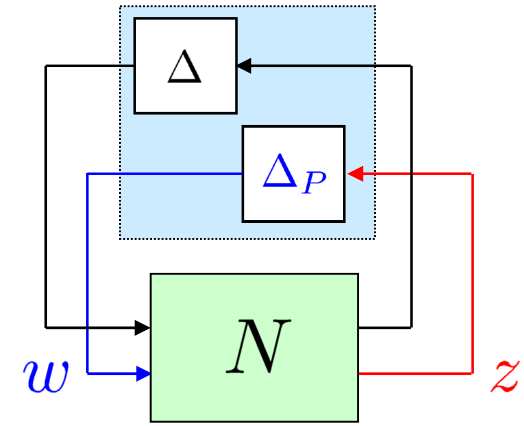
$$\|F_u(N,\Delta)\|_\infty = \|N_{22} + \underline{N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}}\|_\infty$$





Robust Performance [SP05, p. 317]



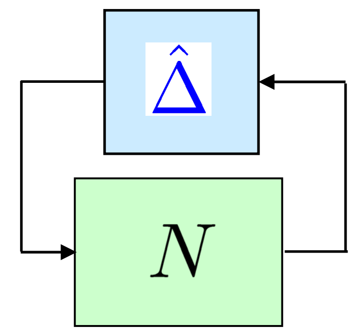


Fictitious "Performance Block" Δ_P

$$\hat{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix} \quad \Delta_P : \text{Full Block}$$

Structured Uncertainties

【SP05, Theorem 8.7】 (p. 317)



$$\text{RP} \Leftrightarrow \|F_u(N, \Delta)\|_\infty < 1, \ {}^\forall \|\Delta\|_\infty \le 1$$

$$\Leftrightarrow \mu_{\hat{\Delta}}(N(j\omega)) < 1, \ {}^\forall \omega$$

What's μ ?

Main Loop Theorem [SP05, p. 317]

【Theorem】 $\mu_{\hat{\Delta}}(N) < 1 \;\Leftrightarrow\; \det(I - N\hat{\Delta}) \neq 0, \;{}^{\forall}\hat{\Delta}, \bar{\sigma}(\hat{\Delta}) \leq 1$

Proof:

$$\begin{aligned}
&\det(I - N\hat{\Delta}) \\
&= \det\left(I - \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}\begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix}\right) = \det\begin{bmatrix} I - N_{11}\Delta & -N_{12}\Delta_P \\ -N_{21}\Delta & I - N_{22}\Delta_P \end{bmatrix} \\
&= \det(I - N_{11}\Delta)\cdot\det(I - (N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12})\Delta_P) \\
&= \det(I - N_{11}\Delta)\cdot\det(I - F_u(N,\Delta)\Delta_P) \\
&\neq 0, \;{}^{\forall}\Delta, {}^{\forall}\Delta_P \;\; \|\Delta\|_\infty \leq 1
\end{aligned}$$

$$\mu_{\hat{\Delta}}(N) < 1 \;\Leftrightarrow\; \begin{cases} \det(I - N_{11}\Delta) \neq 0, \;{}^{\forall}\Delta, \bar{\sigma}(\Delta) \leq 1 \\ \det(I - F_u(N,\Delta)\Delta_P) \neq 0, \;{}^{\forall}\Delta_P, {}^{\forall}\Delta \end{cases}$$

$$\Leftrightarrow \begin{cases} \mu_{\Delta}(N_{11}) < 1, \;{}^{\forall}\omega & \text{Robust Stability} \\ \mu_{\Delta_P}(F_u(N,\Delta)) < 1, \;{}^{\forall}\omega & \text{Robust Performance} \end{cases}$$

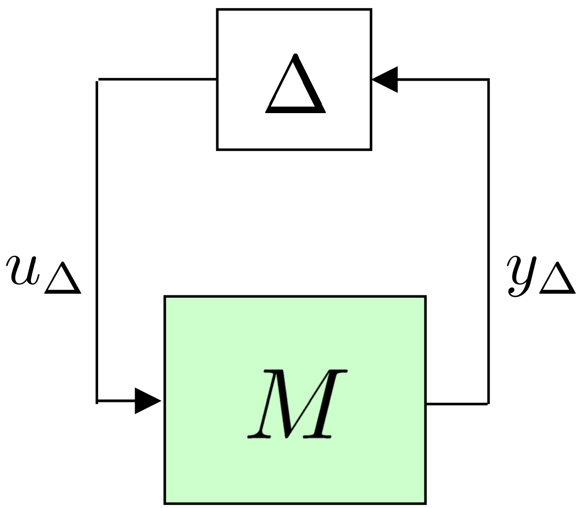
Stability of Closed-loop System [SP05, pp. 301-303]

Unstructured Uncertainties

$$\Delta = \begin{bmatrix} \Delta \end{bmatrix} = \begin{bmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{bmatrix}$$

Full-block

$$\|\Delta\|_\infty \leq 1$$



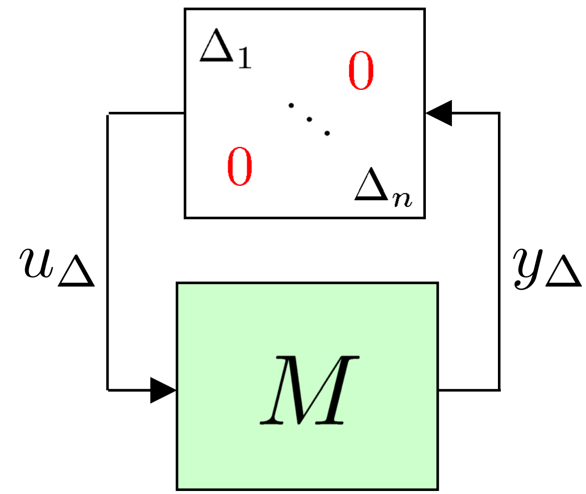
$$\|M\|_\infty < 1 \;\; (\bar{\sigma}(M) < 1)$$

($\because$ Small Gain Theorem)

Structured Uncertainties

$$\Delta = \mathrm{diag}\{\Delta_i\} = \begin{bmatrix} \Delta_1 & & & 0 \\ & \ddots & & \\ & & \Delta_i & \\ 0 & & & \ddots \end{bmatrix}$$

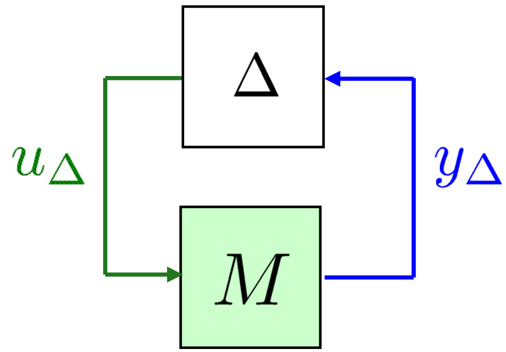
$$\|\Delta\|_\infty \leq 1$$



$$\mu_\Delta(M) < 1$$

Stability of $M\Delta$-structure [SP05, p. 301]

$$\begin{cases} y_\Delta = M u_\Delta \\ u_\Delta = \Delta y_\Delta \end{cases}$$



$$\Delta = \mathrm{diag}\{\Delta_i\} = \begin{bmatrix} \Delta_1 & & 0 \\ & \ddots & \\ & \Delta_i & \\ 0 & & \ddots \end{bmatrix}$$

Structured

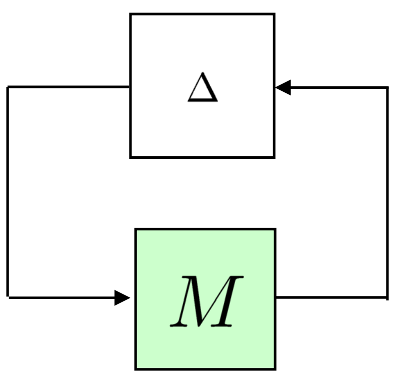
$$y_\Delta = M\Delta y_\Delta \ \rightarrow \ (I - M\Delta) y_\Delta = 0$$

$$\det(I - M\Delta) = 0 \ ?$$

$$\Delta = 0 : \det(I - M \cdot 0) = 1 \neq 0$$ (Stable)

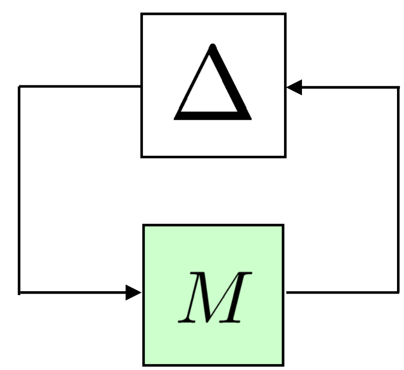
$$\det(I - M\Delta) \neq 0$$

$$\det(I - M\Delta) = 0 \ !$$



Stable

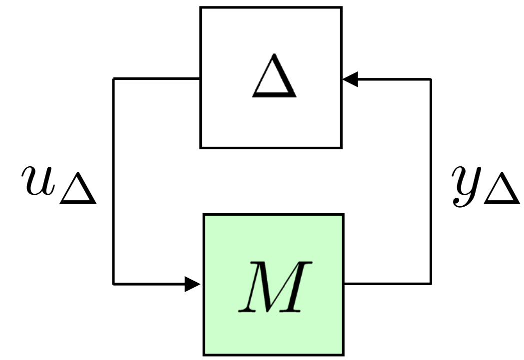
$\Delta \rightarrow$ Large



Unstable !

Stability of $M\Delta$-structure (Cont'd)

$$\Delta = \begin{bmatrix} \Delta_1 & & & 0 \\ & \ddots & & \\ & & \Delta_i & \\ 0 & & & \ddots \end{bmatrix} \qquad \Delta = \begin{bmatrix} \Delta_i & \\ & \Delta_j \end{bmatrix} \qquad \Delta = \begin{bmatrix} \Delta_i & \\ & \Delta_j \end{bmatrix}$$



$$\bar{\sigma}(\Delta) = \max_i \{\bar{\sigma}(\Delta_i)\}$$

- Measure the size of Δ which makes the system unstable, when we make the structured uncertainty Δ large gradually.

- Find the smallest structured Δ satisfying $\det(I - M\Delta) = 0$ and measure the size $\bar{\sigma}(\Delta)$.

$$\mu(M)^{-1} = \min_{\Delta}\{\bar{\sigma}(\Delta) | \det(I - M\Delta) = 0 \text{ for Structured } \Delta\}$$

Structured Singular Value (SSV) [SP05, p. 306]

【SP05, Definition 8.1】 (p. 308)

For $M \in \mathcal{C}^{n\times n}$, μ_Δ is defined

$$\mu_\Delta(M) := \frac{1}{\min\{\bar{\sigma}(\Delta)|\Delta \in \boldsymbol{\Delta},\ \det(I - M\Delta) = 0\}}$$

unless no $\Delta \in \boldsymbol{\Delta}$ makes $I - M\Delta$ singular,

in which case $\mu_\Delta(M) := 0$.

$$\mu_\Delta(M) < 1 \iff \text{The } \Delta : \bar{\sigma}(\Delta) > 1$$
$$\iff \det(I - N\Delta) \neq 0,\ {}^\forall\Delta,\ \bar{\sigma}(\Delta) \leq 1$$

Stable in Large Δ "Good" ⟷ Optimal Control Small

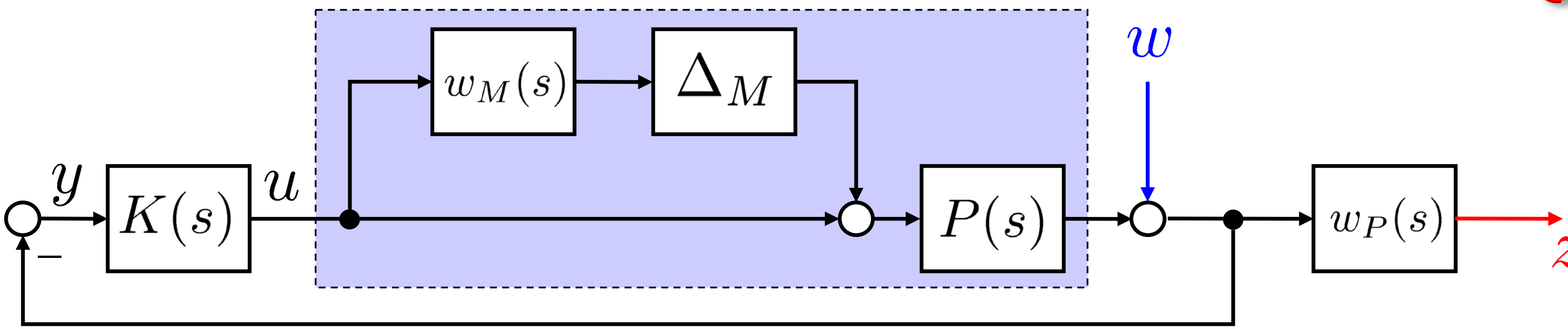
Unstable in Small Δ "Bad" ⟷ Optimal Control Large

Inverse ⟹ $\mu \to$ "Minimize"

[Ex.] (i) $\mu = 2.0 \quad (> 1)$ ⟷ $\bar{\sigma}(\Delta) = \frac{1}{2} = 0.5 \ (< 1)$ Small

(ii) $\mu = 0.66\cdots (< 1)$ ⟷ $\bar{\sigma}(\Delta) = \frac{3}{2} = 1.5 \ (> 1)$ Large

Robust Performance in SISO Systems [SP05, p. 322]



Robust Performance: $\mu_{\hat{\Delta}}(N) < 1$

$$N = \begin{bmatrix} -w_M T & -w_M KS \\ w_P PS & w_P S \end{bmatrix}$$

$$\mu \begin{bmatrix} -w_M T & -w_M KS \\ w_P SP & w_P S \end{bmatrix} \begin{matrix} \times P \\ \times P^{-1} \end{matrix} = \mu \begin{bmatrix} -w_M T & -w_M T \\ w_P S & w_P S \end{bmatrix} = |w_M T| + |w_P S| < 1, \ \forall \omega$$

$$(T = PK(1+PK)^{-1} = PKS)$$

Structured Uncertainty

for $\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}$, $\mu \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \mu \begin{bmatrix} m_{11} & dm_{12} \\ \frac{1}{d} m_{21} & m_{22} \end{bmatrix}$, $\mu \begin{bmatrix} t & t \\ s & s \end{bmatrix} = |t| + |s|$

MATLAB Command mussv

Compute Bounds on Structured Singular Value μ

$$\max_{U\in\mathcal{U}} \rho(MU) \le \mu_\Delta(M) \le \min_{D\in\mathcal{D}} \bar{\sigma}(DMD^{-1})$$

```
[bounds, muinfo] = mussv( M, blk, option )
```

Input argument

- M Frequency transfer matrix
- blk Structure of uncertainty

Output argument

- bounds Upper and lower bounds of μ
- muinfo Information on these bounds

Information on the bound can be extracted by the command mussvextract

Option

- ‘a’ These bounds are computed to the maximal accuracy of LMI solver
- ‘f’ Only the upper bound is roughly computed
- ‘U’ Only the upper bound is computed
- ‘x’ The lower bound is roughly computed

Why is μ used?

μ ➡ \ mu ➡ mu ➡ 無

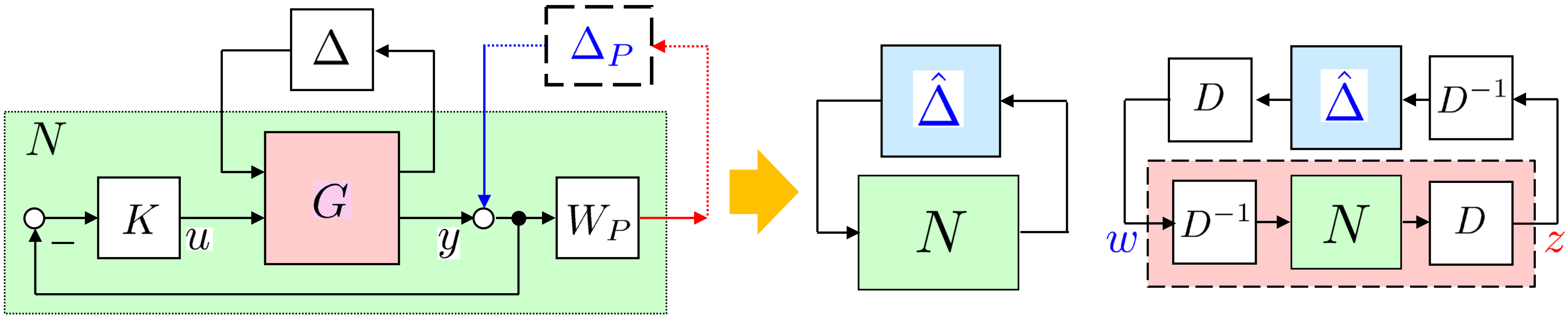
J. C. Doyle

森 政弘

鈴木大拙「無」

μ-Analysis for Robust Performance [SP05, p. 305]

Given a controller K,



Robust Performance

$$\mu_{\hat{\Delta}}(N) < 1$$

Upper Bound and Lower Bound

$$\max_{U \in \mathcal{U}} \rho(NU) \leq \mu_{\hat{\Delta}}(N) \leq \min_{D \in \mathcal{D}} \bar{\sigma}(DND^{-1})$$

$$\Rightarrow \min_{D \in \mathcal{D}} \bar{\sigma}(DND^{-1}) < 1$$

$D-$ Scaled Maximum Singular Value

(Since it is computationally hard to obtain μ, the bounds are employed)

Beyond Loop Shaping

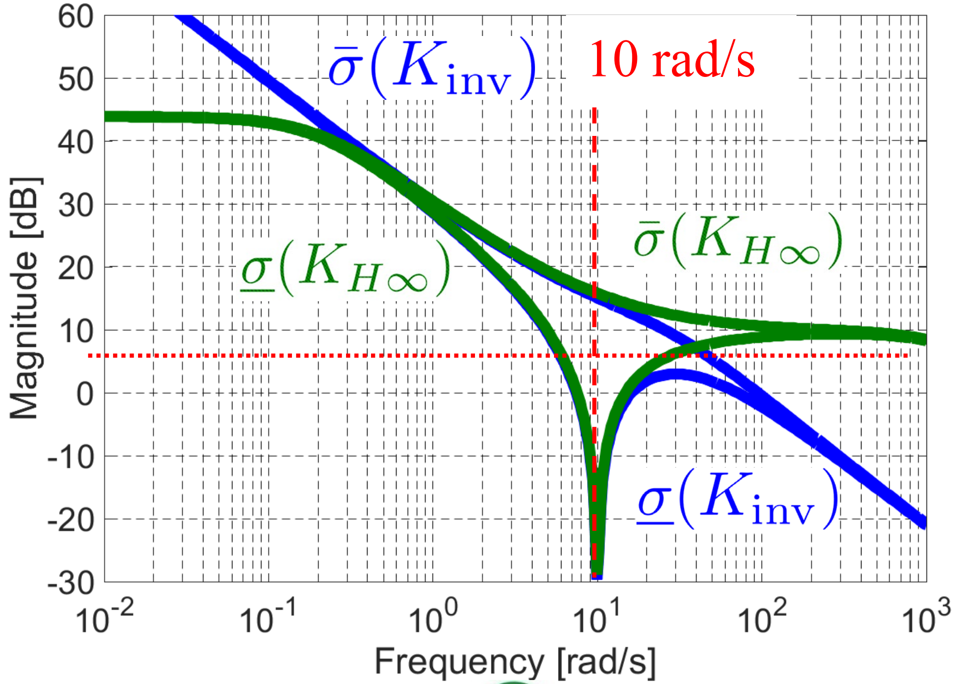
$\Rightarrow$ μ -analysis $\mu_{\hat{\Delta}}(N) < 1$

Spinning Satellite: Controllers K_{inv} and $K_{H\infty}$

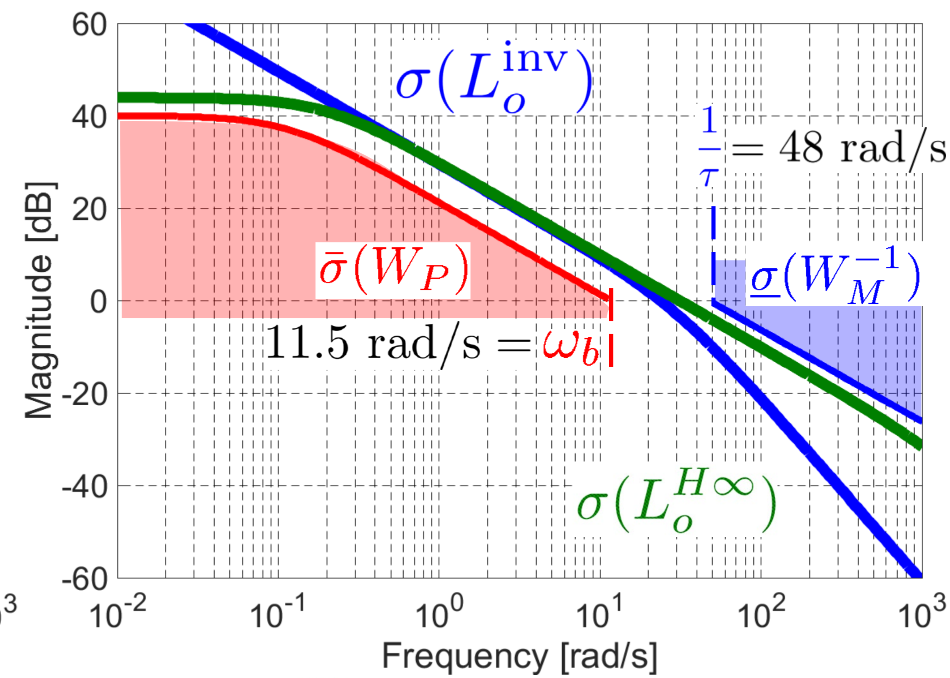
Inverse-based Controller K_{inv}
$$K_I(s) = P^{-1}(s)\begin{bmatrix} \frac{900}{s(s+30)} & 0 \\ 0 & \frac{900}{s(s+30)} \end{bmatrix}$$

H_∞ Controller $K_{H\infty}$ hinfsyn

Controller

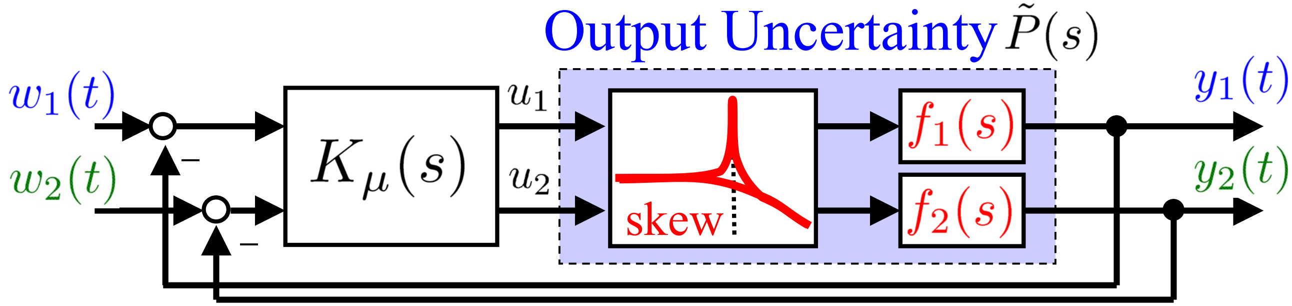


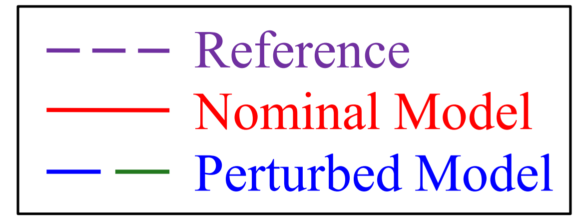
Loop Shaping



- **NS** Youla Parameterization (H_∞ Controller)
- **NP** $\|W_P S\|_\infty < 1$
- **RS** $\|W_M T\|_\infty < 1$

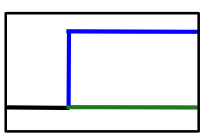
Spinning Satellite: Time Responses for K_{inv} and $K_{H\infty}$





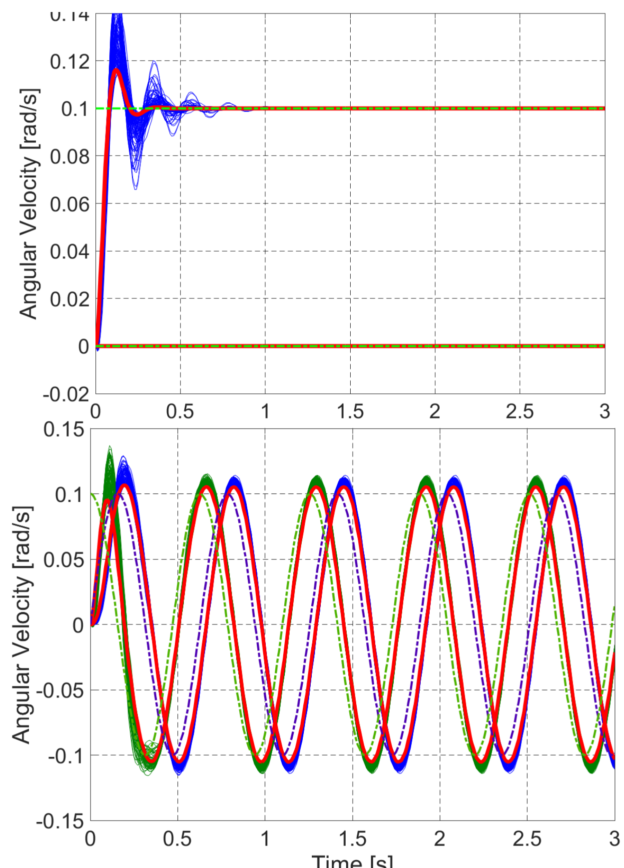
Inverse-based Controller K_{inv} | H_∞ Controller $K_{H\infty}$

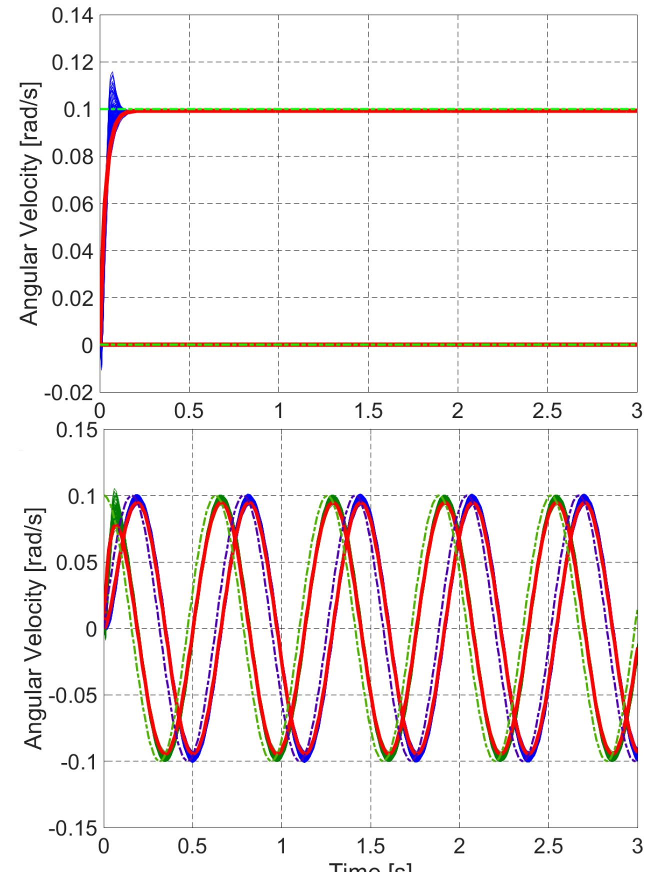
$$w(t) = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$$



$$w(t) = \begin{bmatrix} 0.1\sin(\omega t) \\ 0.1\cos(\omega t) \end{bmatrix}$$

$$\omega = 10 \ \text{rad/s}$$

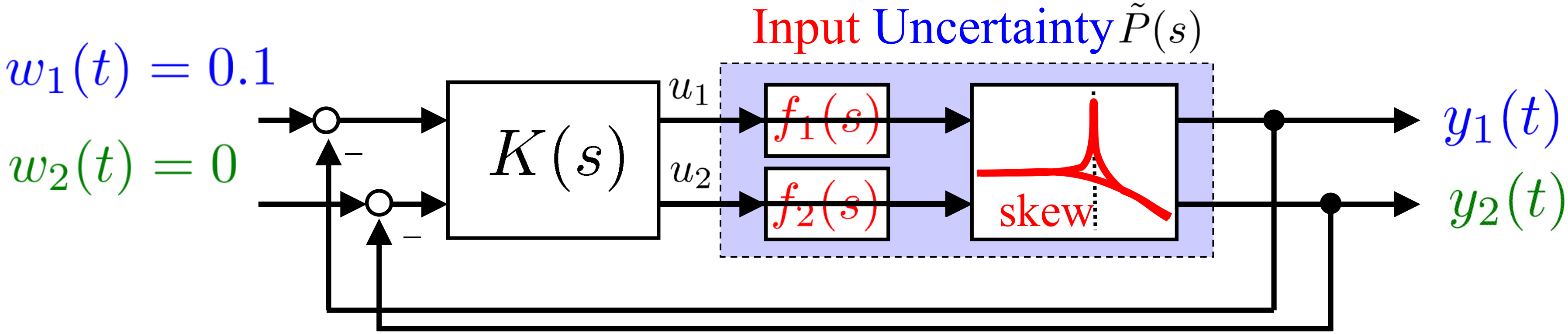




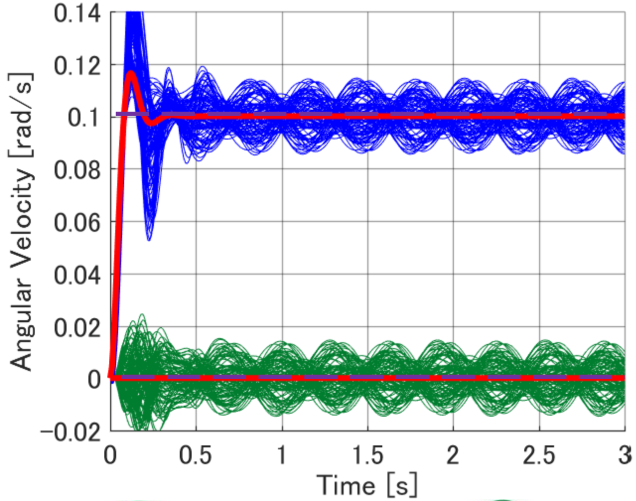
NS, NP, RS (inverse-based controller K_{inv})

NS, NP, RS (H_∞ controller $K_{H\infty}$)

Spinning Satellite: Time Responses for Input Uncertainty

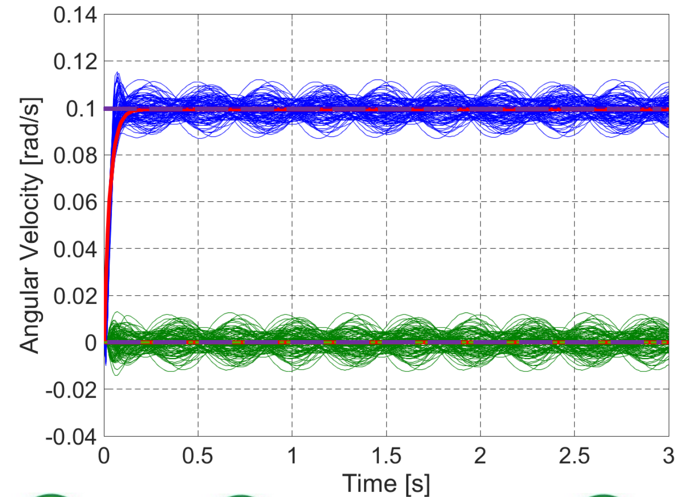


Inverse-based Controller K_{inv}



NS, NP, RS(Input), RP

H_∞ Controller $K_{H\infty}$



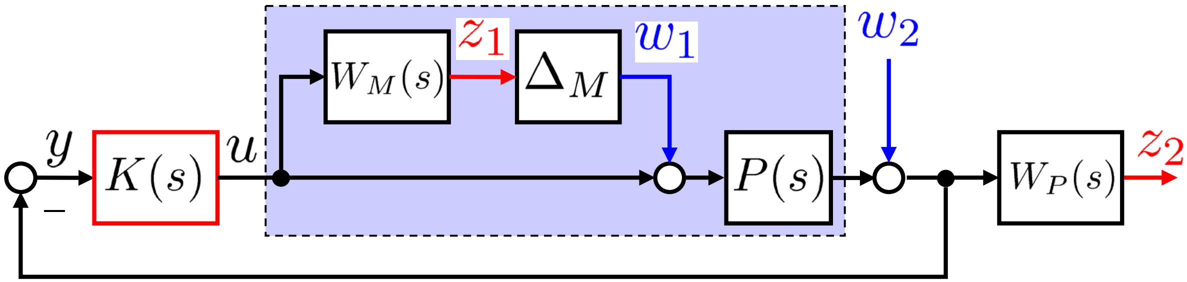
NS, NP, RS(Input), RP

Reference (dashed)
Nominal Model (solid)
Perturbed Model (dashed)

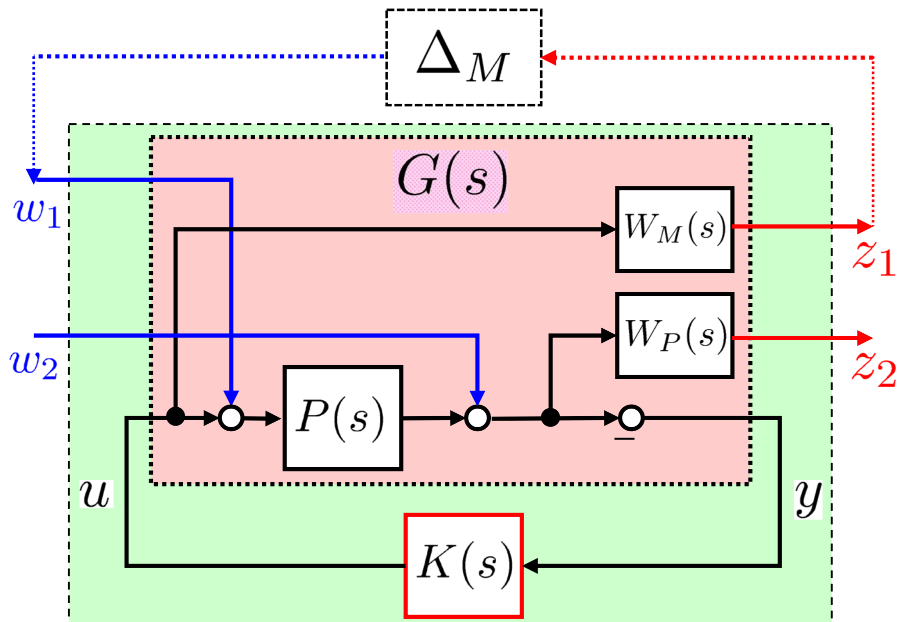
Beyond Loop Shaping

Spinning Satellite: Generalized Plant

Input Uncertainty



Generalized Plant

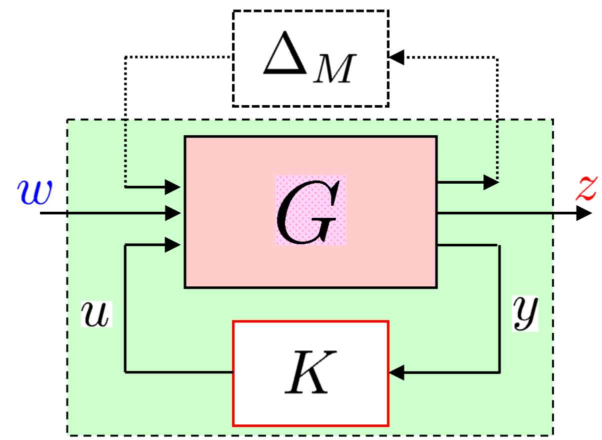


$$z = F_l(G,K)w$$

$$F_l(G,K) = \begin{bmatrix} -W_M T_i & -W_M K S_o \\ W_P S_o P & W_P S_o \end{bmatrix}$$

MATLAB Command

```
%Generalized Plant%
systemnames = 'Pnom WP WM';
inputvar = '[w1(2); w2(2); u(2)]';
outputvar = '[WM;WP;-w2-Pnom]';
input_to_Pnom = '[u+w1]';
input_to_WP = '[w2+Pnom]';
input_to_WM = '[u]';
G = sysic;
%with Structured Uncertainty%
unc1 = ultidyn('unc1',[1 1]);
unc2 = ultidyn('unc2',[1 1]);
unc = [unc1 0; 0 unc2];
Gunc = lft(unc,G);
```



$$\Delta_M = \begin{bmatrix} \delta_{M1} & 0 \\ 0 & \delta_{M2} \end{bmatrix}$$

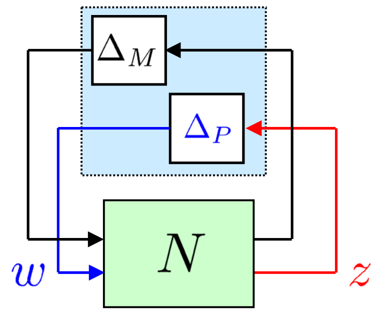
Structured Uncertainty

Spinning Satellite: μ-Analysis of K_{inv} and $K_{H\infty}$

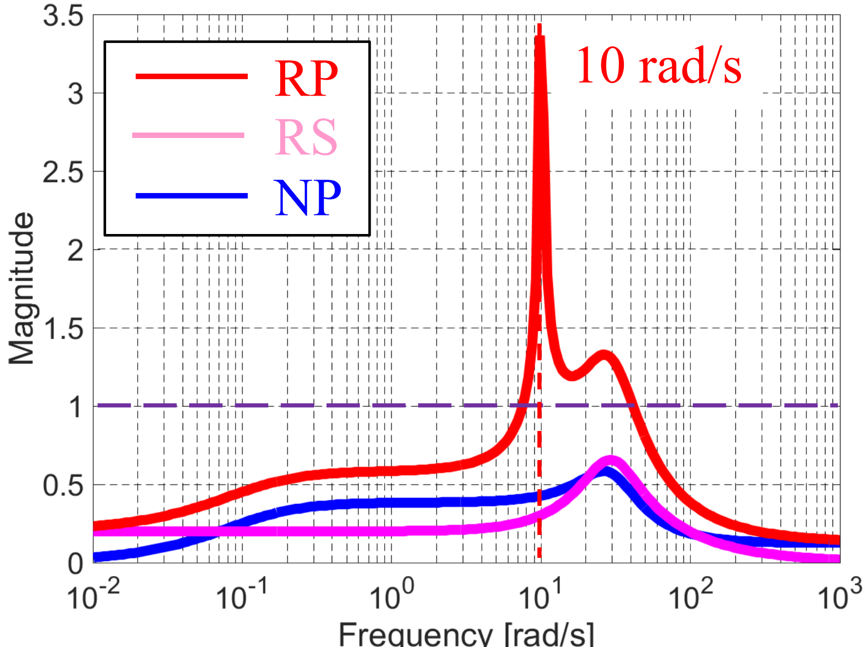
mussv

$$N = F_l(G, K)$$

$$= \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} -W_M T_i & -W_M K S_o \\ W_P S_o P & W_P S_o \end{bmatrix}$$



Inverse-based Controller K_{inv}

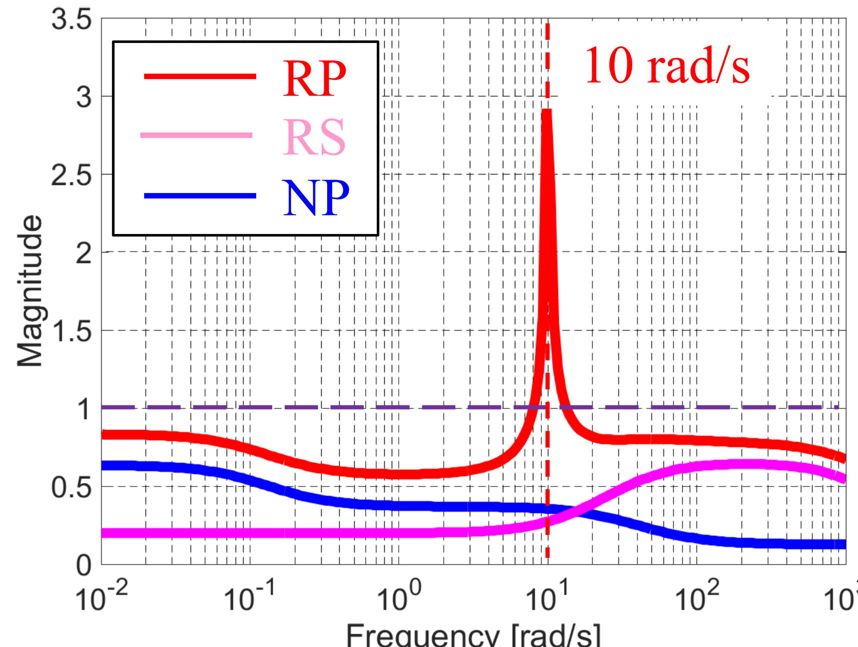


$\mu_{\Delta_P}(N_{22})$ = 0.5847 NP

$\mu_{\Delta_M}(N_{11})$ = 0.6569 RS(Input)

$\mu_{\hat{\Delta}}(N)$ = 3.3644 RP

H_∞ Controller $K_{H\infty}$

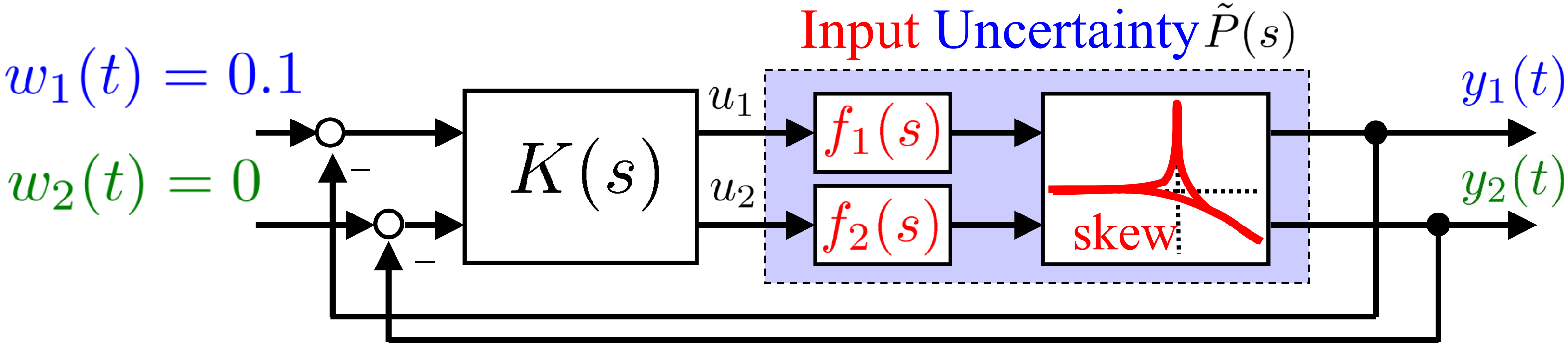


$\mu_{\Delta_P}(N_{22})$ = 0.6331 NP

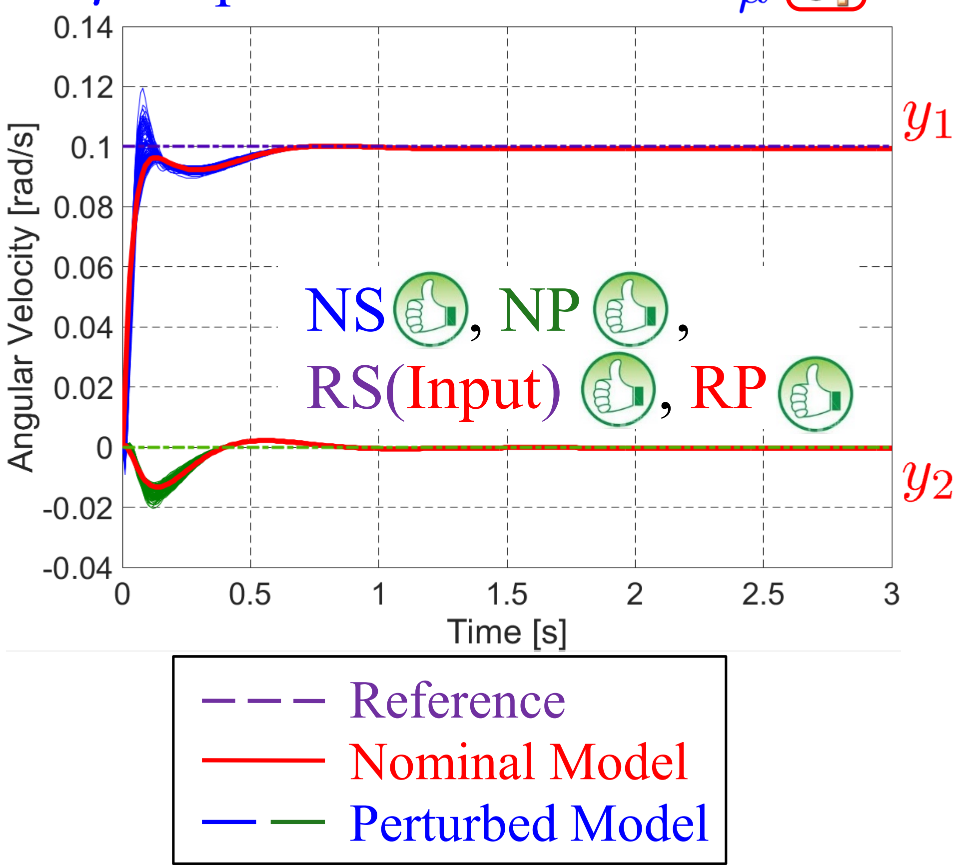
$\mu_{\Delta_M}(N_{11})$ = 0.6416 RS(Input)

$\mu_{\hat{\Delta}}(N)$ = 2.9159 RP

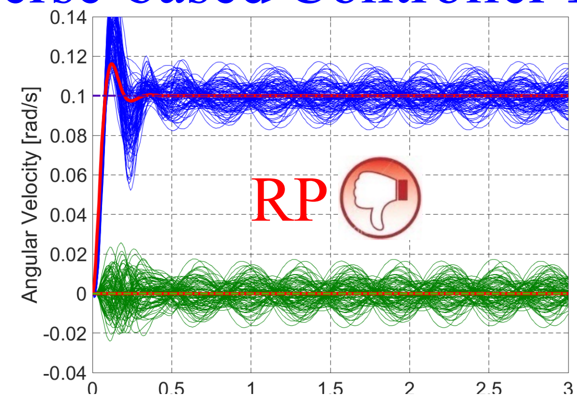
Spinning Satellite: Time Responses for Input Uncertainty



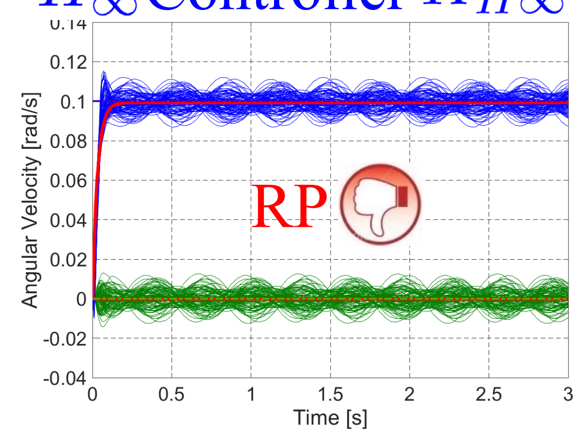
μ-Optimal Controller K_μ



Inverse-based Controller K_{inv}



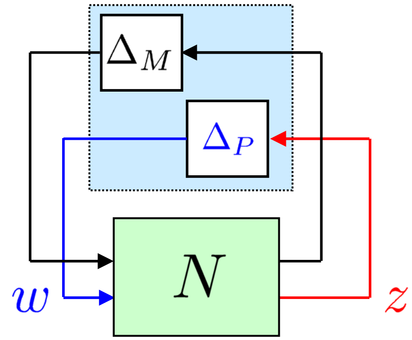
H_∞ Controller $K_{H\infty}$

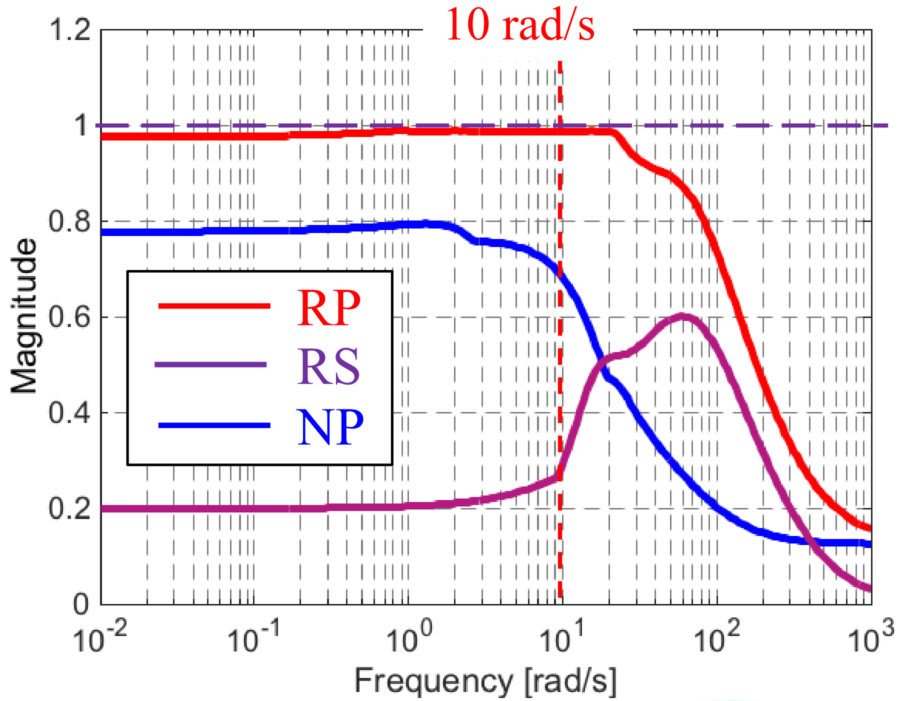


Spinning Satellite: μ -Analysis for K_μ

mussv

$$N = F_l(G, K_\mu) = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} -W_M T_i & -W_M K S_o \\ W_P S_o P & W_P S_o \end{bmatrix}$$





$\mu_{\Delta_P}(N_{22})$ = 0.7936 NP

$\mu_{\Delta_M}(N_{11})$ = 0.6001 RS

$\mu_{\hat{\Delta}}(N)$ = 0.9898 RP

MATLAB Command

```
Blk_unc = [1 1;1 1];
Blk_per = [2 2];
Blk = [Blk_unc; Blk_per];
%%%
w = logspace(-2,2,200);
Nf = frd(N,w);
%%% mu for NP %%%
Nnp = Nf(3:4,3:4);
[MuBnds,MuInfo] = mussv(Nnp,Blk_per,'c');
muNP = MuBnds(:,1);
[muNPinf,muNPw] = norm(muNP,inf);
muNPinf
 %%% mu for RS %%%
Nrs = Nf(1:2,1:2);
[MuBnds,MuInfo] = mussv(Nrs,Blk_unc,'c');
muRS = MuBnds(:,1);
[muRSinf,muRSw] = norm(muRS,inf);
muRSinf
%%% mu for RP %%%
[MuBnds,MuInfoRP] = mussv(Nf,Blk,'c');
muRP = MuBnds(:,1);
[muRPinf,muRPw] = norm(muRP,inf);
muRPinf
%%%
figure; sigma(muNP,muRS,muRP)
```

μ-Synthesis and DK-iteration

MATLAB Command dksyn

[k, cl, bnd, info] = dksyn(G, nmeas, ncont, option)

Input argument

G	Generalized Plant
nmeas	Number of measurement outputs
ncont	Number of control inputs

Output argument

k	Controller
cl	Closed-loop system which consists of G and K
bnd	Upper bound of μ
info	Information of Iteration

option

Note: use dksynOptions/dkitopt to create option

FrequencyVector (Default: []) i.e. automatically the frequency range is chosen

InitialController (Default: [])

AutoIter (Default: 'on')

DisplayWhileAutoIter (Default: 'off')

StartingIterationNumber (Default: '1')

NumberOfAutoIteration (Default: '10')

AutoIterSmartTerminateTol (Default '0.005')

MixedMU (Default: 'off')

AutoScalingOrder (Default: '5')

AutoIterSmartTerminate (Default: 'on')

μ-synthesis [SP05, p. 328]

$$\min_K \max_\omega \mu_{\hat{\Delta}}(N(j\omega))$$

"At present there is no direct method to synthesize a μ-optimal controller"

$$\Rightarrow \min_K \max_\omega \min_{D_\omega \in \mathcal{D}} \bar{\sigma}\left(D_\omega N(j\omega) D_\omega^{-1}\right)$$

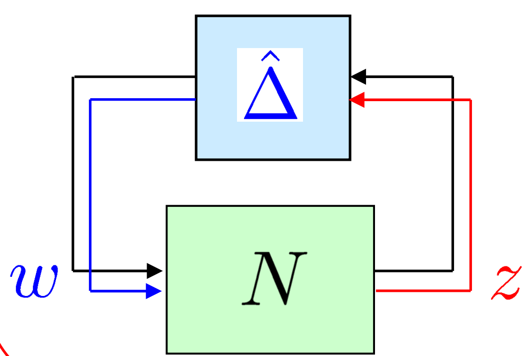
$$\Rightarrow \min_K \left(\min_{D \in \mathcal{D}} \| D N(j\omega) D^{-1} \|_\infty \right)$$

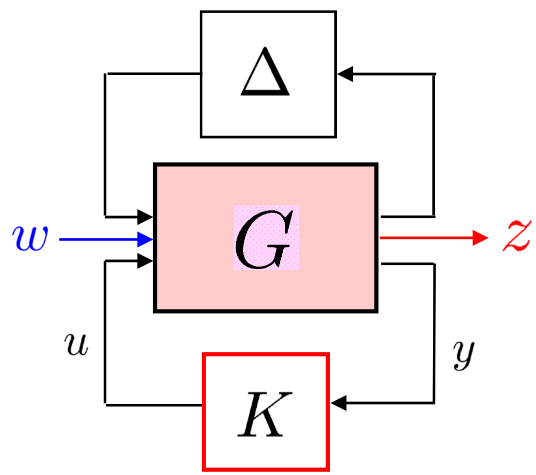
Scaled H_∞ Control

(D-Step)

μ-analysis

Structured Uncertainty

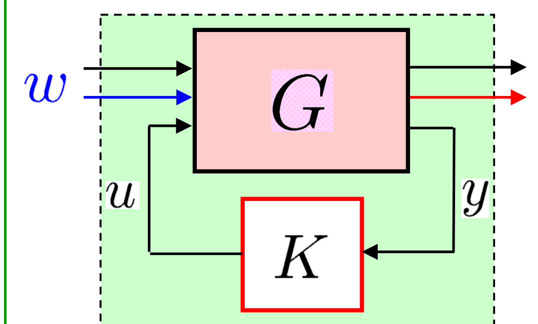




DK-iteration

(K-Step)

H_∞-synthesis



$$z = F_l(G, K)w$$

$$N = F_l(G, K)$$

The iterations may converge to a local optimum. However, practical experience suggests that the method work well in most cases.

Computational Complexity

R. D. Brattz, R. M. Young, J. C. Doyle and M. Morari,
"Computational Complexity of μ Calculation,"
IEEE Trans. Automatic Control, **39**-5, 1000-1002, 1994

NP Hard　　Uncertainty ➡ Complexity

40 years of robust control: 1978-2018

2014 American Control Conference

G. J. Balas, J. C. Doyle, P. Gahinet, K. Glover, A. K. Packard, P. Seiler and R. S. Smith

- Robust Control is a beautiful theory
  … based on a solid principles
- Yet, it has practical limitations
  Produces monolithic(一枚岩), black-box controllers
  Controller complexity tends to be high
- … that make it difficult to apply

But, tools are available to apply Robust Control methodology to real-world applications

Epilogue

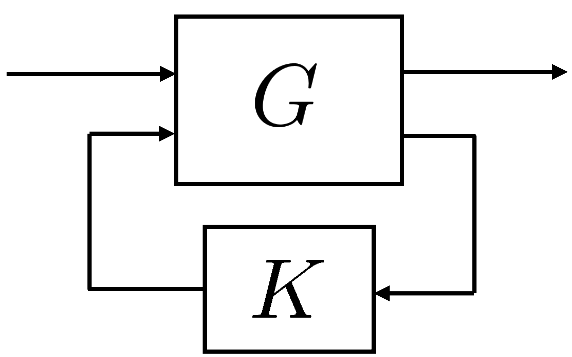
∼ 20th Century

From SISO system to MIMO system

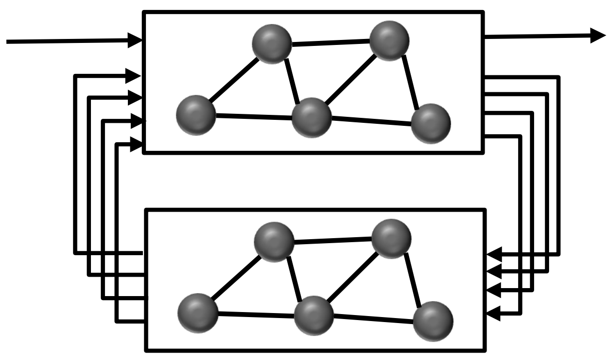
21st Century ∼

From Single-Agent system to Multi-Agent system

Robust control of uncertain system



Cooperative/Distributed control of network system



IEEE CSS Video Clip Contest

2017
Second Place

Click

https://www.youtube.com/watch?v=QUIcQ8PkJpM

System Level Synthesis

1940's –1950's→

Tough and Strong

1960's –1970's→

Smart and Intelligent

1980's –1990's→

Tough and Smart

2000's – 2010's →

Tough, Smart and Elegant

K. Zhou, J. Doyle and K. Glover, *Robust and Optimal Control*, Prentice Hall, 1996

J. Doyle, N. Matni, Y.-S. Wang, J. Anderson and S. Low, "System Level Synthesis: A Tutorial," CDC 2017

7. Robust Performance

- 7.1 Robust Performance [SP05, Sec. 7.6, 8.3, 8.4, 8.10]
- 7.2 Structured Singular Value μ [SP05, Sec. 8.5, 8.6, 8.8, 8.11]
- 7.3 μ-Analysis and Synthesis [SP05, Sec. 7.6, 8.9, 8.10, 8.11]

Reference:

[SP05] S. Skogestad and I. Postlethwaite, ***Multivariable Feedback Control; Analysis and Design,*** Second Edition, Wiley, 2005.

8. Design Example

8.1 HiMAT: H_∞ Control
(Highly Maneuverable Aircraft Technology)

Reference:

[SP05] S. Skogestad and I. Postlethwaite,
Multivariable Feedback Control; Analysis and Design,
Second Edition, Wiley, 2005.

[SLH81] M. G. Safonov, A. J. Laub and G. L. Hartmann,
Feedback Properties of Multivariable Systems:
The Role and Use of the Return Difference Matrix,
IEEE Transactions on Automatic Control, Vol. 26, No. 1, pp. 47-65, 1981.

[ES84] M. B. Evans and L. J. Schilling,
The Role of Simulation in the Development and Flight Test of the HiMAT Vehicle,
NASA Technical Memorandum 84912, 1984.

[CS98] R. Y. Chiang and M. G. Safonov,
Robust Control Toolbox User's Guide,
The MathWorks, 1998.

Structured Robust Stability

【SP05, Theorem 8.6】(p. 314) RS for Block-diagonal Perturbations

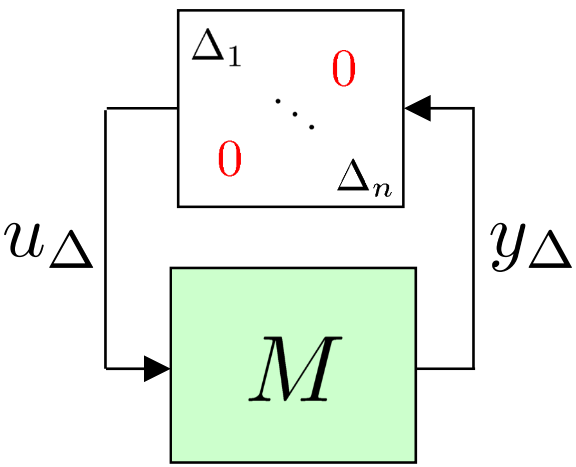
Assume that the nominal system $M(s)$ and the perturbations $\Delta(s)$ are stable. Then the $M\Delta$ -system is stable for all perturbation with

$$\bar{\sigma}(\Delta(j\omega)) \leq 1, \ {}^{\forall}\omega \qquad \left[\bar{\sigma}(\Delta(j\omega)) \leq \frac{1}{\beta}, \ {}^{\forall}\omega \right]$$

if and only if

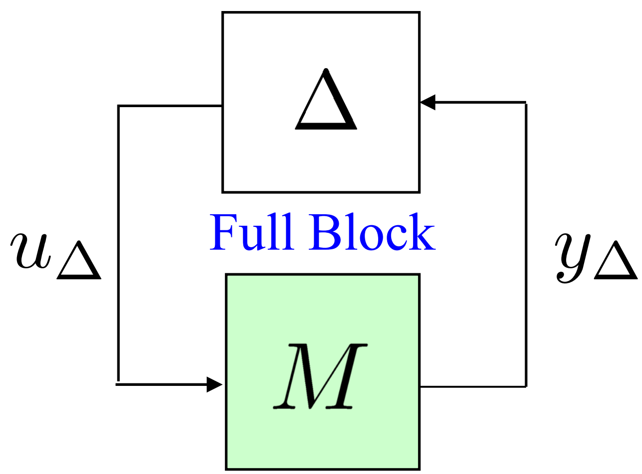
$$\mu_{\Delta}(M) < 1, \ {}^{\forall}\omega \qquad \left[\ \mu_{\Delta}(M) < \beta, \ {}^{\forall}\omega \ \right]$$

Structured



$\mu_{\Delta}(M) < 1 \ {}^{\forall}\omega$

Unstructured



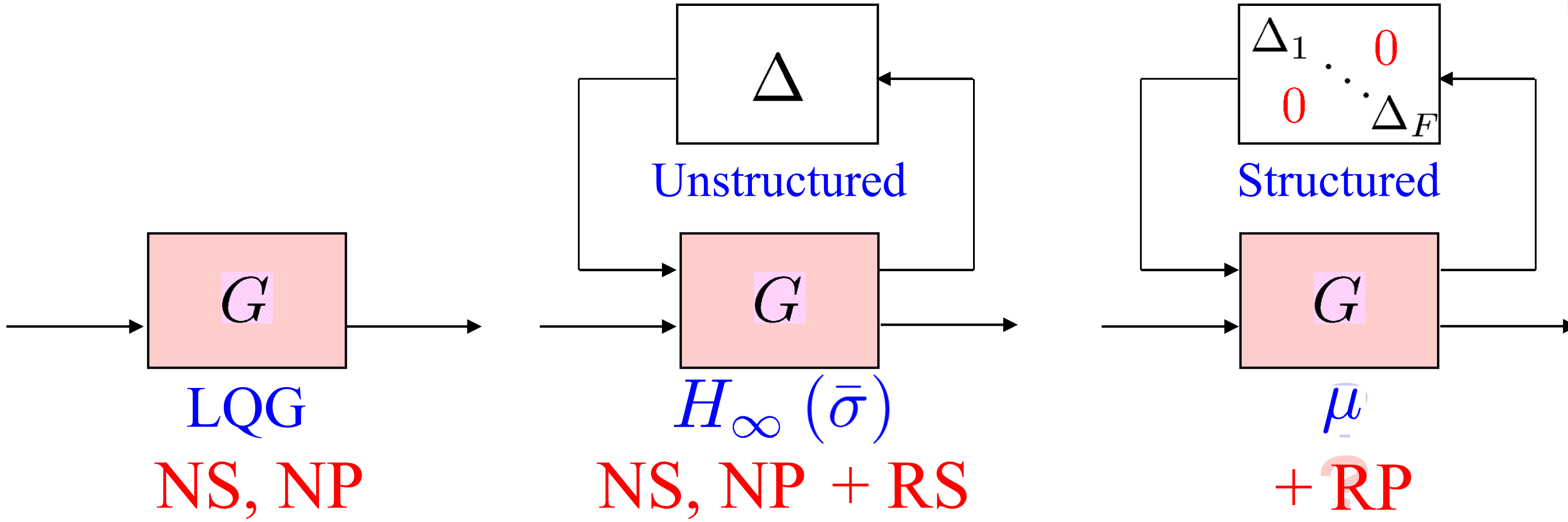
$\mu_{\Delta}(M) = \bar{\sigma}(M) < 1 \ {}^{\forall}\omega$

Beyond
Singular Value σ

Structured
Singular Value μ

Structured Uncertainty

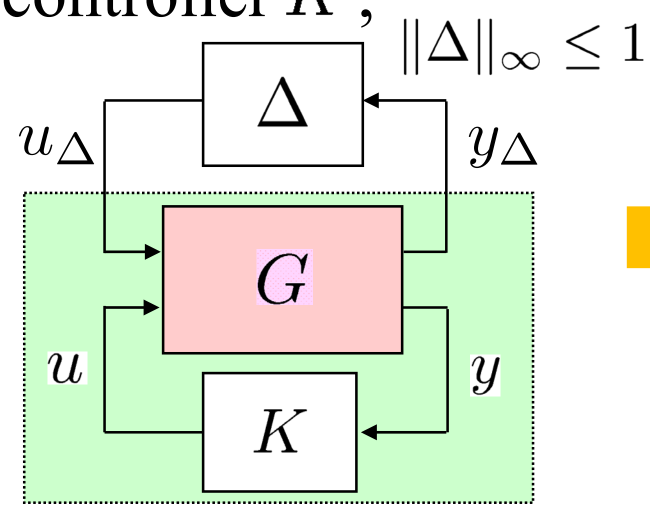
Block Diagonal

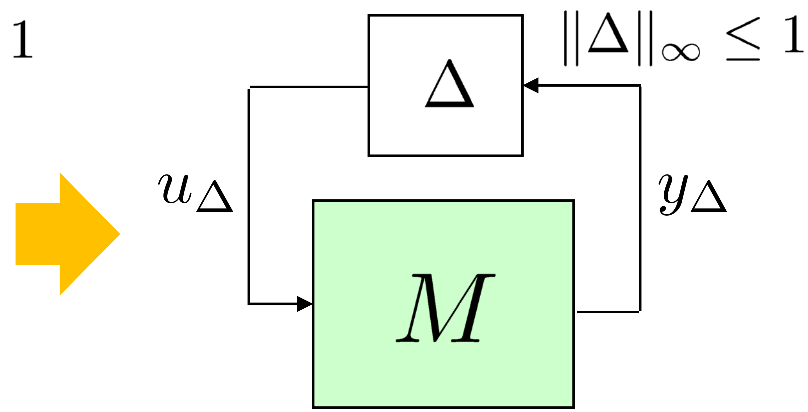


Loop Shaping Info. on (non-)interaction

Feedback System with Structured Uncertainties

Given a controller K,





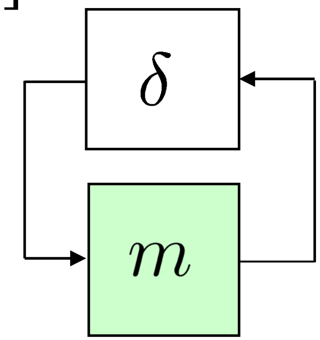
$M\Delta$ -structure

Mathematical Properties of μ [SP05, pp. 309-312]

μ of a Scalar

$$m \in \mathcal{C}, \quad \delta \in \mathcal{C}$$

$$(1 - m\delta) = 0 \Rightarrow |\delta| = \frac{1}{|m|}$$

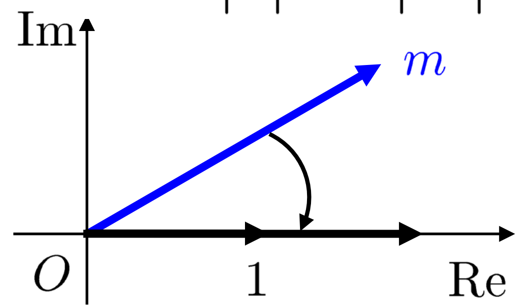


$$\mu_\delta(m) = \frac{1}{\min\{|\delta| \; |1 - m\delta = 0\}} = |m|$$

$$m = re^{j\phi}, \quad \delta = \frac{1}{r}e^{-j\phi}$$

$$(1 - m\delta) = 0 \Rightarrow |\delta| = \frac{1}{|r|} = \frac{1}{|m|}$$

$$\mu_\delta(m) = |m|$$



$$m = 0$$

$$\mu_\Delta(m) = 0$$

Full Block

$$M \in \mathcal{C}^{n\times n}, \quad \Delta = \left[\,\Delta\,\right] \in \mathcal{C}^{n\times n}$$

$$M = U\Sigma V^H, \quad \Delta = (1/\sigma_1)v_1 u_1^H$$

$$\mu_\Delta(M) = \bar{\sigma}(M)$$

Equal to the maximal singular value in the absence of the structure

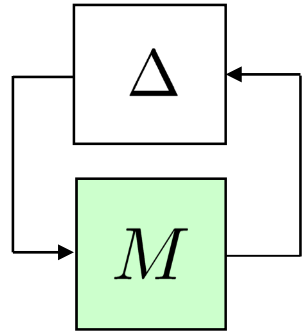
$$\det(I - M\Delta) = \det(I - U\Sigma V^H v_1 u_1^H/\sigma_1) = 1 - u_1^H U\Sigma V^H v_1/\sigma_1 = 0$$

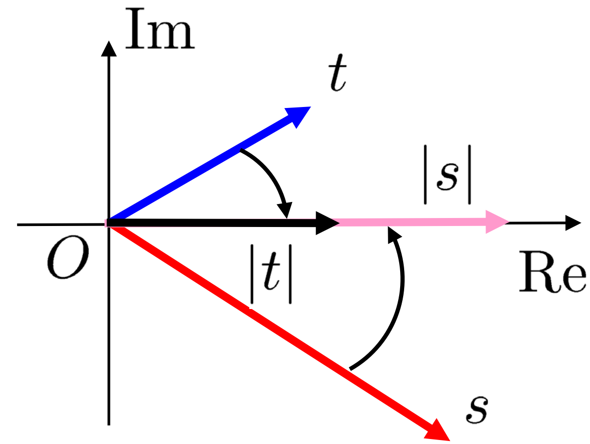
u_1, v_1 : 1st columns of U, V, $\sigma_1 = \bar{\sigma}(M)$ $\because \det(I - AB) = \det(I - BA)$

Mathematical Properties of μ [SP05, pp. 309-312]

Special Case of 2×2 Matrices [SP05, Ex. 8.7, pp. 312]

$$M = \begin{bmatrix} t & t \\ s & s \end{bmatrix}, \Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}$$





$$\mathrm{rank}(M) = 1$$

$$M\Delta = \begin{bmatrix} t & t \\ s & s \end{bmatrix} \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} = \begin{bmatrix} t \\ s \end{bmatrix} \begin{bmatrix} \delta_1 & \delta_2 \end{bmatrix} =: \tilde{M}\tilde{\Delta}$$

$$\det(I - M\Delta) = \det(I - \tilde{M}\tilde{\Delta}) = \det(I - \tilde{\Delta}\tilde{M}) = 1 - t\delta_1 - s\delta_2$$

$$|\delta_1| = |\delta_2| = \frac{1}{|t| + |s|}; \quad 1 - t\delta_1 - s\delta_2 = 0 \quad \Rightarrow \quad \mu_\Delta(M) = |t| + |s|$$

Scaling [SP05, Ex. 8.16, pp. 313]

$$\boldsymbol{\Delta} = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} \quad \Rightarrow \quad \mu \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \mu \begin{bmatrix} m_{11} & dm_{12} \\ \frac{1}{d} m_{21} & m_{22} \end{bmatrix}$$

Mathematical Properties of μ [SP05, pp. 309-312]

[SP05, Ex. 8.5] (p. 307)

$$M = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0.894 & 0.447 \\ -0.447 & 0.894 \end{bmatrix} \begin{bmatrix} 3.162 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix}^H$$

Full Perturbation

$$\Delta = \frac{1}{3.162} \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix} \begin{bmatrix} 0.894 & -0.447 \end{bmatrix} = \begin{bmatrix} 0.200 & -0.100 \\ 0.200 & -0.100 \end{bmatrix}$$

$$\bar{\sigma}(\Delta) = \frac{1}{3.162} = 0.316 \qquad \mu_\Delta(M) = \frac{1}{\bar{\sigma}(\Delta)} = 3.162$$

$$\mu_\Delta(M) = \bar{\sigma}(M) = \sqrt{2 \cdot |2|^2 + 2 \cdot |-1|^2} = 3.1623$$

Diagonal Perturbation

$$\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix} \in \boldsymbol{\Delta}$$

$$\bar{\sigma}(\Delta) = 0.333 = \frac{1}{\mu_\Delta(M)} \qquad \mu_\Delta(M) = |2| + |-1| = 3$$

Bounds on Structured Singular Value μ mussv

$$\max_{U \in \mathcal{U}} \rho(MU) \le \mu_\Delta(M) \le \min_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1})$$

[SP05, Ex. 8.5] (p. 307)

$$M = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$$

Special case of 2 × 2 Matrices

(i) blk = $\Delta = \begin{bmatrix} \delta_{11} & 0 \\ 0 & \delta_{22} \end{bmatrix}$

Structured (Block Diagonal)

MATLAB Command

```
M = [2 2; -1 -1];
blk = [1 0;1 0]; % structured
[bounds,muinfo] = mussv(M,blk);
bounds
```

Result

bounds = 3.0000 3.0000

$$\mu_\Delta(M) = |2| + |-1| = 3$$

(ii) blk = $\Delta = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix}$

Unstructured (Full Block)

MATLAB Command

```
M = [2 2; -1 -1];
blk = [2 2]; % unstructured
[bounds,muinfo] = mussv(M,blk);
bounds
```

Result

bounds = 3.1623 3.1623

$$\mu_\Delta(M) = \bar{\sigma}(M) = \sqrt{10} = 3.1623$$

Mathematical Properties of μ [SP05, pp. 309-312]

【SP05, Lemma 8.5】 (p. 309)

$$\mu_\Delta(M) = \max_{\Delta \in \mathbf{B\Delta}} \rho(M\Delta) \qquad \mathbf{B\Delta} = \{\Delta \in \mathbf{\Delta} : \bar{\sigma}(\Delta) < 1\}$$

where $\rho(A) := \max_i |\lambda_i(A)|$ denotes spectral radius of matrix A .

Properties

2. $\mu_\Delta(M) = \rho(M)$ for $\mathbf{\Delta} = \{\delta I : \delta \in \mathcal{C}\}$ (Repeated Scalar Perturbation)
3. $\mu_\Delta(M) = \bar{\sigma}(M)$ for $\mathbf{\Delta} = \mathcal{C}^{n\times n}$ (Full-block Perturbation)
4. $\rho(M) \le \mu_\Delta(M) \le \bar{\sigma}(M)$, $\{\delta I : \delta \in \mathcal{C}\} \subset \mathbf{\Delta} \subset \mathcal{C}^{n\times n}$
6. $D\Delta = \Delta D$, $D \in \mathcal{D}, \Delta \in \mathbf{\Delta}$.

   Then, $\mu_\Delta(DM) = \mu_\Delta(MD)$, $\mu_\Delta(M) = \mu_\Delta(DMD^{-1})$

   $\mathcal{U} = \{U \in \mathbf{\Delta} : UU^H = I_n\}$ (Unitary Matrix)

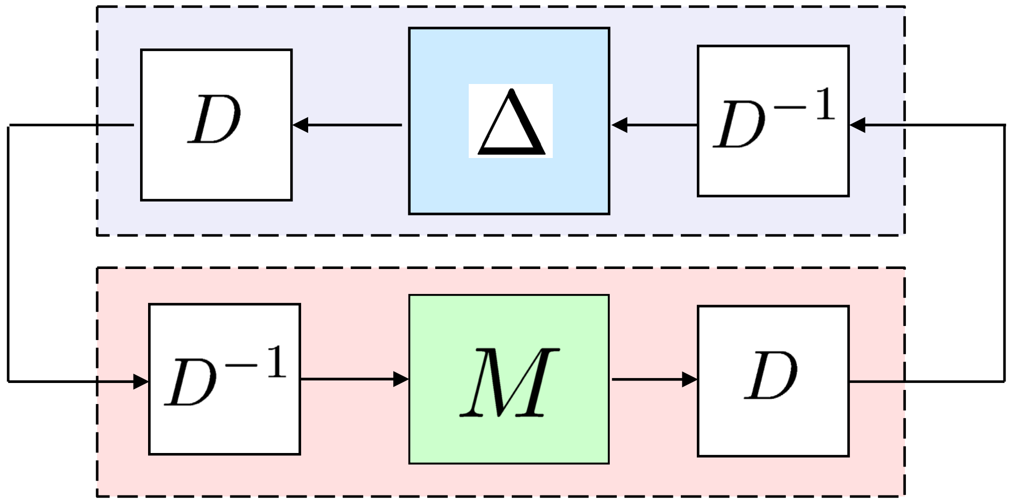
   $\mathcal{D} = \{\mathrm{diag}(d_1 I_{m_1}, \cdots, d_{F-1} I_{m_{F-1}}, I_{m_F}) : d_i \in \mathcal{R}, d_i > 0\}$

7., 8. Upper Bound and Lower Bound (Reduction of Conservatism)

$$\max_{U\in\mathcal{U}} \rho(MU) \le \mu_\Delta(M) \le \min_{D\in\mathcal{D}} \bar{\sigma}(DMD^{-1})$$

Computation of the Upper Bound [SP05, p. 336]

$$\mu_\Delta(M) \le \min_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1}) < \beta$$



[SP05, Ex. 12.4] (p. 478)

$$\bar{\sigma}(DMD^{-1}) < \beta \Leftrightarrow (DMD^{-1})^H DMD^{-1} < \beta^2 I$$

$$\Leftrightarrow M^H D^H DM - \beta^2 D^H D < 0$$

$$\inf_{D \in \mathcal{D}} \min_{\beta} \{\beta : M^H DM - \beta^2 D < 0\}$$

It may be solved using LMI (Linear Matrix Inequality)

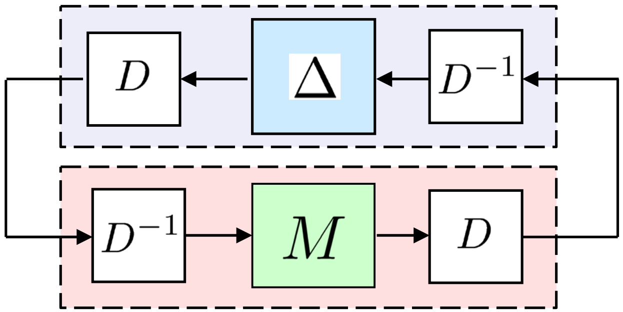
Upper Bound and Lower Bound

$$\max_{U\in\mathcal{U}}\rho(MU) \le \mu_\Delta(M) \le \min_{D\in\mathcal{D}}\bar{\sigma}(DMD^{-1})$$

[SP05, Ex. 8.15] (p. 312)

$$M = \begin{bmatrix} t & t \\ s & s \end{bmatrix}, \Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}$$

Same uncertainty



New M : DMD^{-1}

Lower Bound $U = \text{diag}\{e^{j\phi}, 1\}$

$$\max_{U\in\mathcal{U}}\rho(MU) = \max_\phi \left|\text{tr}\left(\begin{bmatrix} te^{j\phi} & t \\ se^{j\phi} & s \end{bmatrix}\right)\right| = \max_\phi |te^{j\phi}+s| = |t|+|s| = \mu_\Delta(M)$$

Upper Bound $D = \text{diag}\{d, 1\}$ $\left(\|A\|_F = \sqrt{\sum_i \sigma_i^2(A)}\right)$

$$\bar{\sigma}(DMD^{-1}) = \|DMD^{-1}\|_F = \sqrt{|t|^2+|dt|^2+|s/d|^2+|s|^2}$$

$$\min_{D\in\mathcal{D}}\bar{\sigma}(DMD^{-1}) = \min_d \sqrt{|t|^2+|dt|^2+|s/d|^2+|s|^2} = |s|+|t| = \mu_\Delta(M)$$

Structured Uncertainty $\Delta = \text{diag}\{\delta_1 I, \cdots, \delta_S I, \Delta_1, \cdots, \Delta_F\}$

$$2S+F \le 3 \Rightarrow \mu_{\hat{\Delta}}(N) = \min_{D\in\mathcal{D}}\bar{\sigma}(DND^{-1})$$

Upper Bound and Lower Bound [SP05, pp. 309, 310]

Repeated Scalar Complex Perturbation	Full-block Complex Perturbation
$\Delta = \begin{bmatrix} \delta & & 0 \\ & \ddots & \\ 0 & & \delta \end{bmatrix} = \{\delta I : \delta \in \mathcal{C}\}$	$\Delta = \begin{bmatrix} \Delta \end{bmatrix} = \mathcal{C}^{n \times n}$
$\mu_\Delta(M) = \rho(M)$	$\mu_\Delta(M) = \bar{\sigma}(M)$

$$\rho(M) \le \mu_\Delta(M) \le \bar{\sigma}(M)$$

Structured Singular Value

【Theorem】(Upper Bound and Lower Bound)

$$\max_{U \in \mathcal{U}} \rho(MU) \le \mu_\Delta(M) \le \min_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1})$$

$$\mathcal{U} = \{U \in \mathbf{\Delta} : UU^H = I_n\} \qquad \mathcal{D} = \{D : D\Delta = \Delta D\}$$

$UU^H = I_n$: Unitary Matrix

↓

LMI (Linear Matrix Inequality) Computation

Robust Performance in SISO Systems [SP05, p. 281]

NP: Nominal Performance

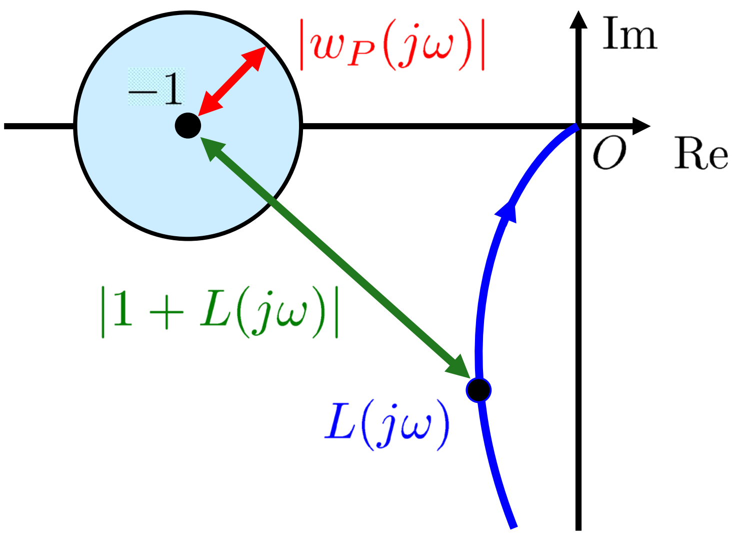
$|w_P S| < 1 \ ^\forall \omega$

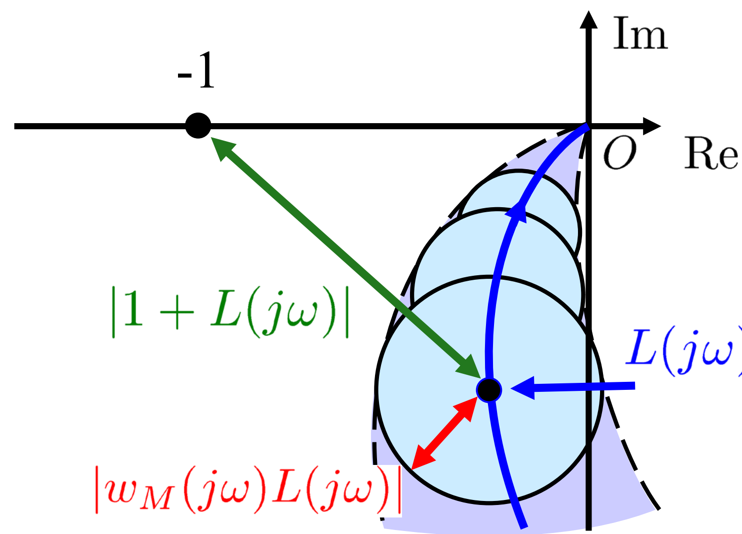
➡ $|w_P| < |1 + L| \ ^\forall \omega$

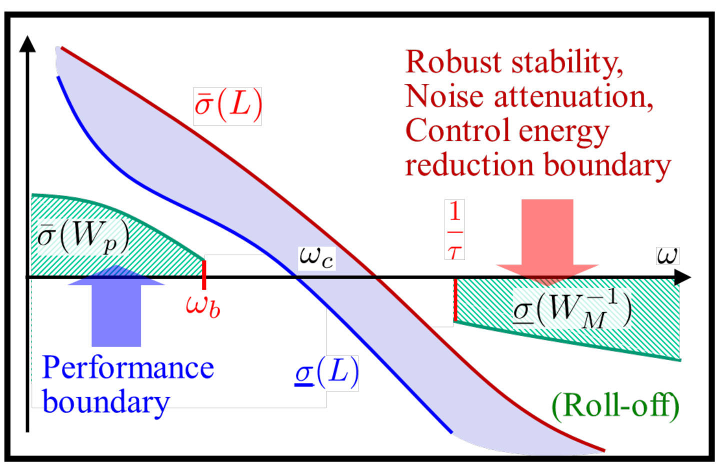
RS: Robust Stability

$|w_M T| < 1 \ ^\forall \omega$

➡ $|w_M L| < |1 + L| \ ^\forall \omega$







Loop Shaping

Robust Performance in SISO Systems [SP05, p. 281]

RP: Robust Performance

Beyond Loop Shaping

$$|w_P S| + |w_M T| < 1, \ ^\forall\omega$$

$$\Rightarrow |w_P| + |W_M L| < |1 + L| \ ^\forall\omega$$