Robust and Optimal Control, Spring 2015

Instructor: Prof. Masayuki Fujita (S5-303B)

A: SISO Feedback Control

A.1 Internal Stability and Youla Parameterization

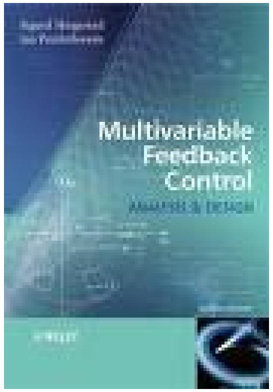
[SP05, Sec. 3.2, 4.1.5, 4.7, 4.8]

A.2 Sensitivity and Feedback Performance

[SP05, Sec. 2.2, 5.2]

A.3 Loop Shaping

[SP05, Sec. 2.4, 2.6]

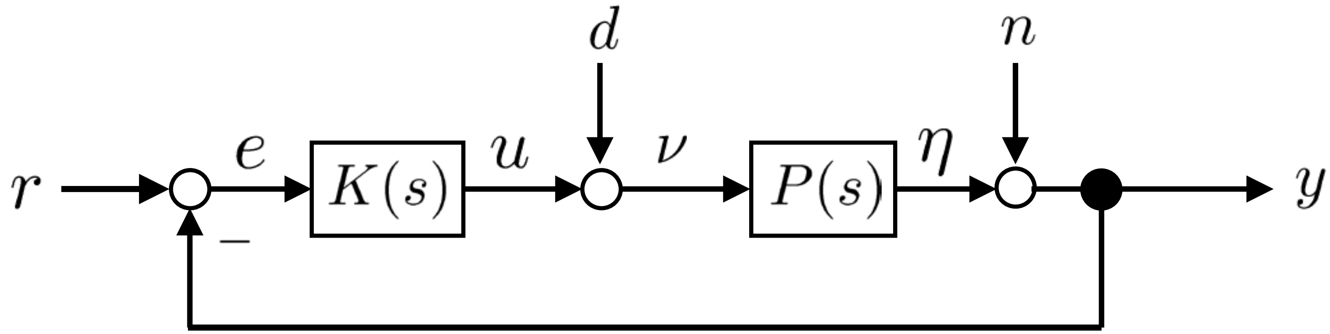


Reference:

[SP05] S. Skogestad and I. Postlethwaite,
Multivariable Feedback Control; Analysis and Design,
Second Edition, Wiley, 2005.

Internal Stability

Gang of Four [AM08, p. 317]



Sensitivity

$$\underset{(n \to y)}{S} = \frac{1}{1+PK}$$

Complementary Sensitivity

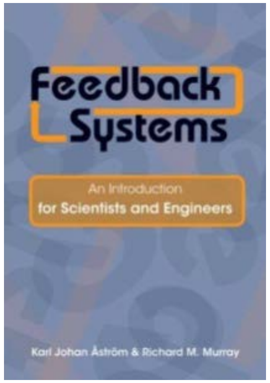
$$\underset{(r \to y)}{T} = \frac{PK}{1+PK}$$

Load Sensitivity

$$\underset{(d \to y)}{PS} = \frac{P}{1+PK}$$

Noise Sensitivity

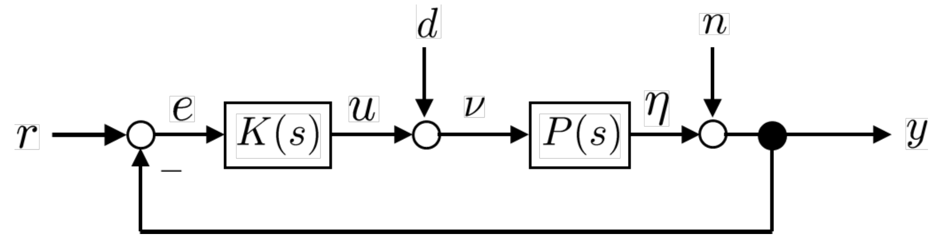
$$\underset{(n \to u)}{KS} = \frac{K}{1+PK}$$



[AM08] K.J. Astrom and R.M. Murray, *Feedback Systems*, Princeton University Press, 2008

Internal Stability

[SP05, Ex. 4.16] (p. 144)



$$P(s) = \frac{s-1}{s+1}\ ,\ K(s) = \frac{k(s+1)}{s(s-1)}\ ,\quad k > 0$$

Sensitivity

$$\underset{(n \to y)}{S} = \frac{s}{s+k}$$

Comp. Sensitivity

$$\underset{(r \to y)}{T} = \frac{k}{s+k}$$

Stable?

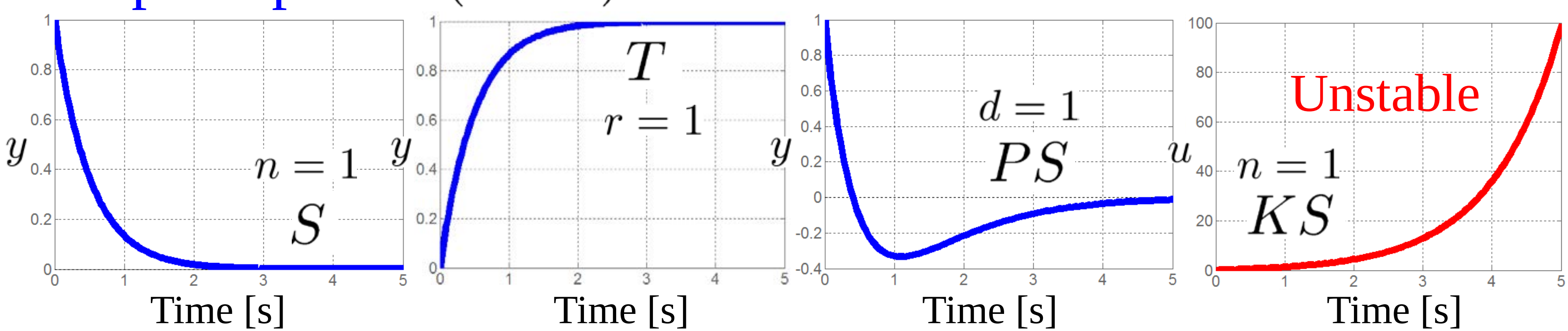
Load Sensitivity

$$\underset{(d \to y)}{PS} = \frac{s(s-1)}{(s+1)(s+k)}$$

Noise Sensitivity

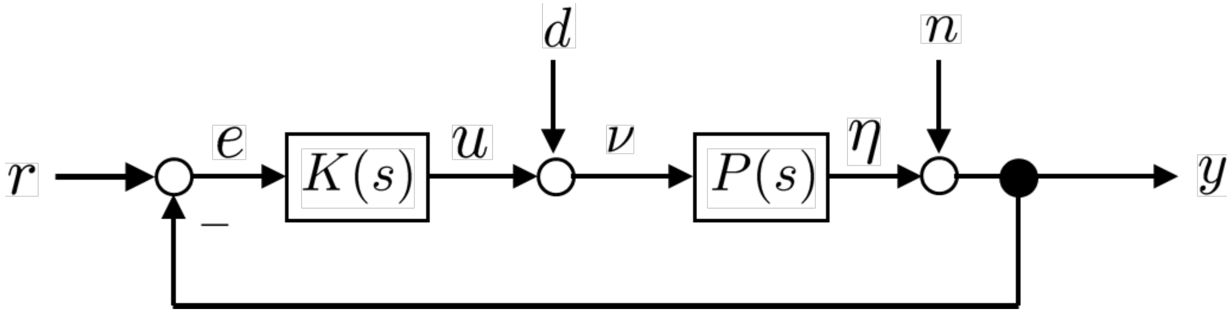
$$\underset{(n \to u)}{KS} = \frac{s+1}{(s-1)(s+k)}$$

Step Response $(k = 2)$



Internal Stability

[SP05, Theorem 4.6] (p. 145)



The feedback system in the above figure is ***internally stable*** if and only if all "Gang of Four (S, T, PS, KS)" are stable

Well-posedness: $1 + P(\infty)K(\infty) \neq 0$

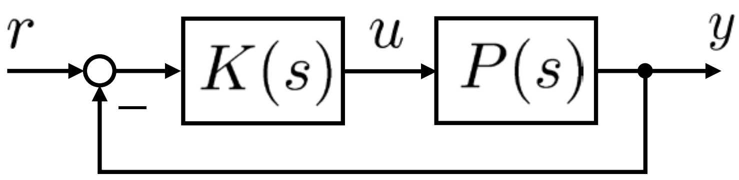
(Gang of Four: well-defined and proper)

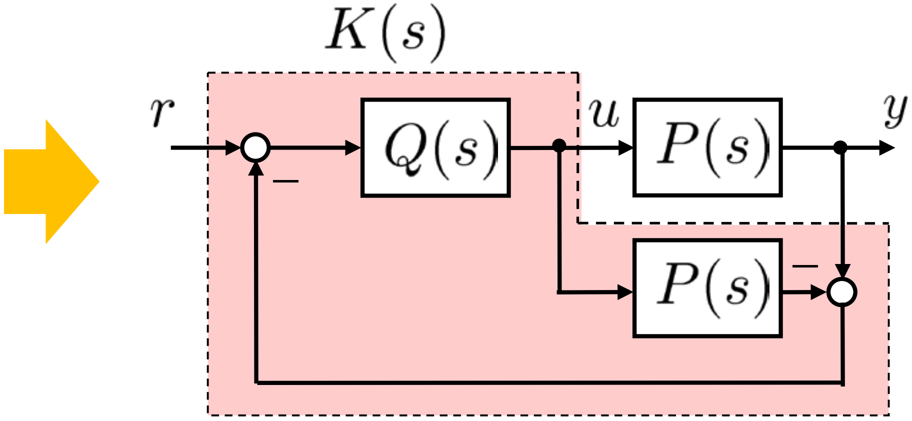
C.A. Desoer and W.S. Chan, Journal of the Franklin Institute, 300 (5-6) 335-351, 1975

C. Desoer

Youla Parameterization (Q Parameterization)

Case 1: Stable Plant $P(s)$ [SP05, p. 148]





Internal Model Control (IMC) Structure

All Stabilizing Controllers: $K(s) = \dfrac{Q(s)}{1 - P(s)Q(s)}$

Q -parameter $Q(s)$: Proper Stable Transfer Function

Gang of Four

$$S = \frac{1}{1+PK} = \underline{1 - PQ} \qquad T = \frac{PK}{1+PK} = \underline{PQ}$$

$$PS = \frac{P}{1+PK} = \underline{P(1 - PQ)} \qquad KS = \frac{K}{1+PK} = \underline{Q}$$

Youla Parameterization

Case 2: Unstable Plant $P(s)$ [SP05, p. 149]

Coprime Factorization [SP05, p. 122]

$$P(s) = \frac{N(s)}{M(s)}$$

Coprime: No common right-half plane(RHP) zeros

$N(s), M(s)$: Proper Stable Transfer Functions

[SP05, Ex. 4.1] $P(s) = \dfrac{(s-1)(s+2)}{(s-3)(s+4)}$

$$\Rightarrow \quad N(s) = \frac{s-1}{s+4}, \quad M(s) = \frac{s-3}{s+2} \qquad (*)$$

Bezout Identity $NX + MY = 1 \longleftrightarrow M(s), N(s)$: Coprime

$X(s), Y(s)$: Proper Stable Transfer Functions

[SP05, Ex.] $M(s), N(s)$: (*) $\Rightarrow$ $X(s) = \dfrac{s+32}{2s+4}, \quad Y(s) = \dfrac{s-16}{2s+8}$

[Ex.] $5x + 3y = 1$ x, y: Integer $\Rightarrow$ $x = -1 - 3q, \quad y = 2 + 5q$

q : Integer

Youla Parameterization

Case 2: Unstable Plants $P(s)$ [SP05, p. 149]

A Stabilizing Controller $K(s) = \dfrac{X(s)}{Y(s)}$ $(Q(s) = 0)$

[SP05, Ex.] $P(s) = \dfrac{(s-1)(s+2)}{(s-3)(s+4)}, \; X(s) = \dfrac{s+32}{2s+4}, \; Y(s) = \dfrac{s-16}{2s+8}$

$$\Rightarrow \quad K(s) = \frac{X(s)}{Y(s)} = \frac{s^2+36s+128}{s^2-14s-32}$$

All Stabilizing Controllers $K(s) = \dfrac{X(s)+M(s)Q(s)}{Y(s)-N(s)Q(s)}$

$$\left(N = P, M = 1, X = 0, Y = 1 \;\Rightarrow\; K = \frac{Q}{1-PQ} \right)$$

Gang of Four

$$S = \underline{M(Y-NQ)} \qquad T = \underline{N(X+MQ)}$$

$$PS = \underline{N(Y-NQ)} \qquad KS = \underline{M(X+MQ)}$$

Affine Functions of Q

Sensitivity and Feedback Performance

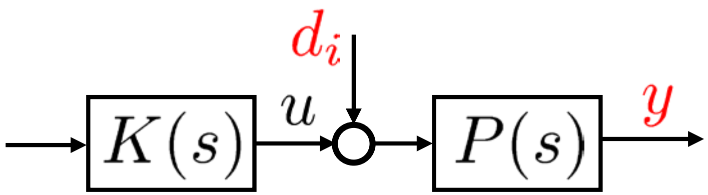
Disturbance Attenuation

Open-loop / **Closed-loop**

$\underline{d_i \to y}$

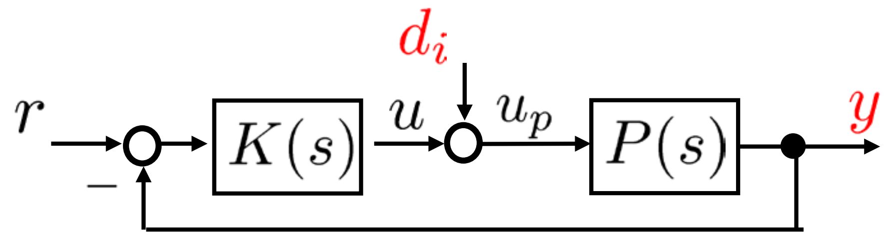
Open-loop:

$$y(s) = P(s)d_i(s)$$



Closed-loop:

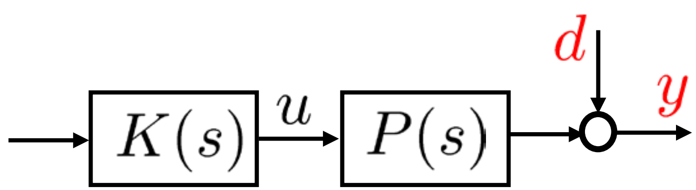
$$y(s) = \frac{1}{1+P(s)K(s)}P(s)d_i(s)$$



$\underline{d \to y}$

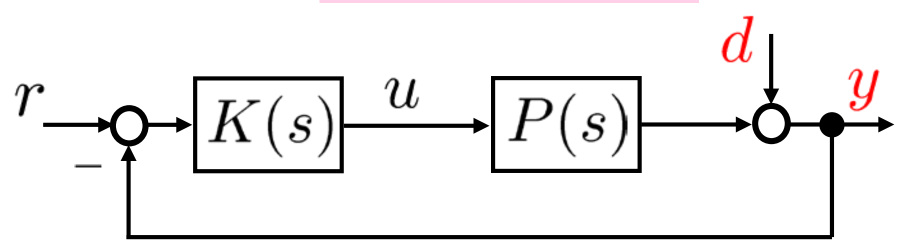
Open-loop:

$$y(s) = d(s)$$



Closed-loop:

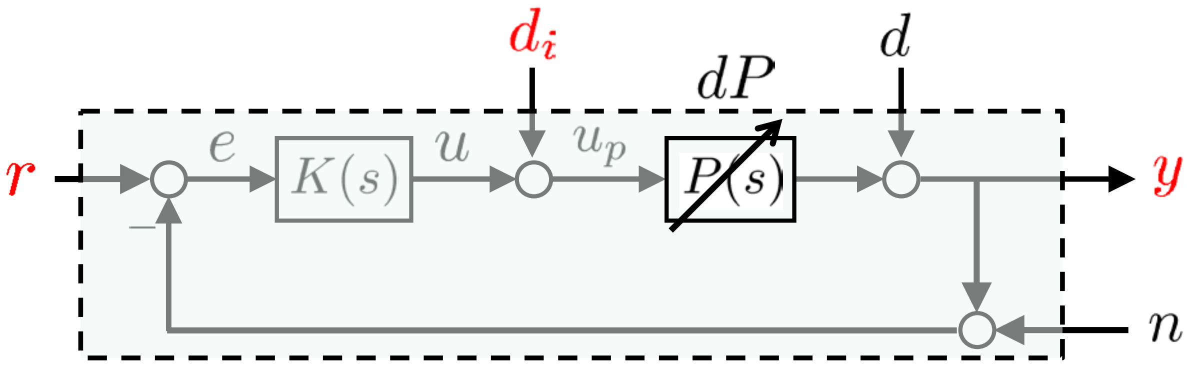
$$y(s) = \frac{1}{1+P(s)K(s)}d(s)$$



$$S(s) = \frac{1}{1+P(s)K(s)}$$: Sensitivity

$|S(j\omega)|$ small: good Feedback Performance

Insensitivity to Plant Variations [SP05, p. 23]



$$G_{yr} = \frac{PK}{1+PK} \quad (r \to y) \qquad \Rightarrow \qquad \frac{dG_{yr}}{G_{yr}} = S\frac{dP}{P}$$

$$\left(\frac{dG_{yr}}{dP} = \frac{K}{(1+PK)^2} = \frac{SPK}{P(1+PK)} = S\frac{G_{yr}}{P}\right)$$

$$G_{yd_i} = \frac{P}{1+PK} \quad (d_i \to y) \qquad \Rightarrow \qquad \frac{dG_{yd_i}}{G_{yd_i}} = S\frac{dP}{P}$$

$$\left(\frac{dG_{yd_i}}{dP} = \frac{1}{(1+PK)^2} = \frac{SP}{P(1+PK)} = S\frac{G_{yd_i}}{P}\right)$$

$|S(j\omega)|$ small : good Feedback Performance

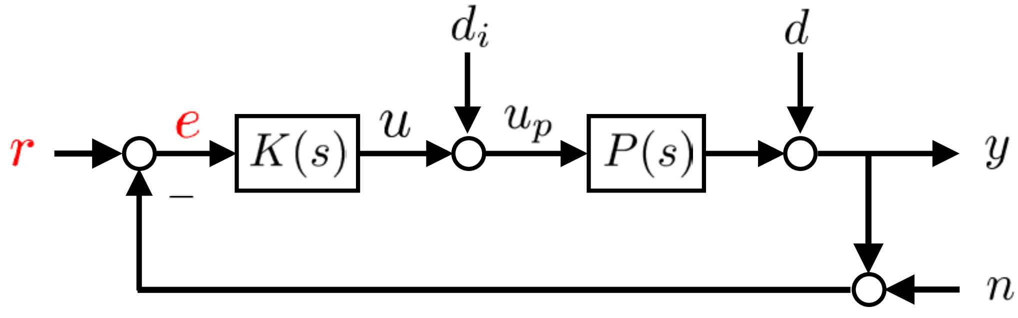
Benefits of Feedback

- Disturbance Attenuation
- Insensitivity to Plant Variations
- Stabilization (Unstable Plant)
- Linearizing Effects
- Reference Tracking

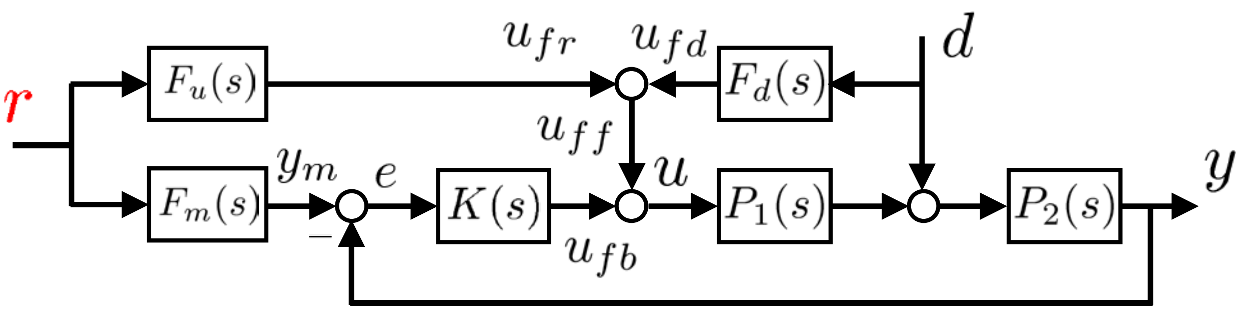
$$G_{er} = \frac{1}{1+PK} = S$$

$(r \to e)$

$|S(j\omega)|$: small



➡ Two-degrees-of-freedom Control Feedback + Feedforward



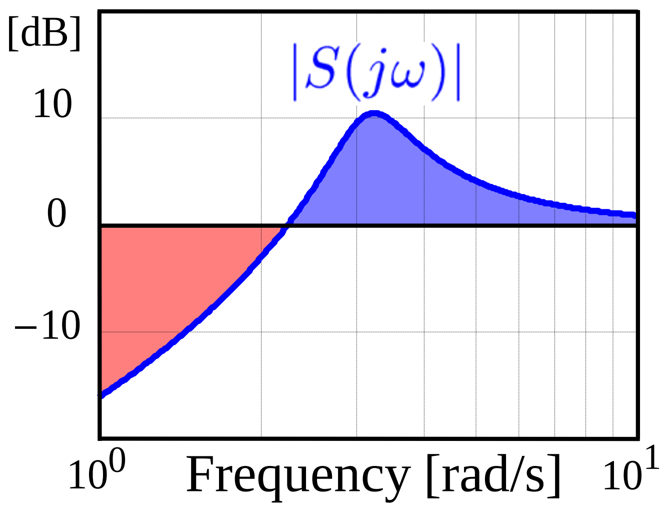
Waterbed Effects [SP05, p. 167]

$$\int_0^\infty \log|S(j\omega)|d\omega = 0$$

$|S| < 1$ $(\log|S| < 0)$

$|S| > 1$ $(\log|S| > 0)$

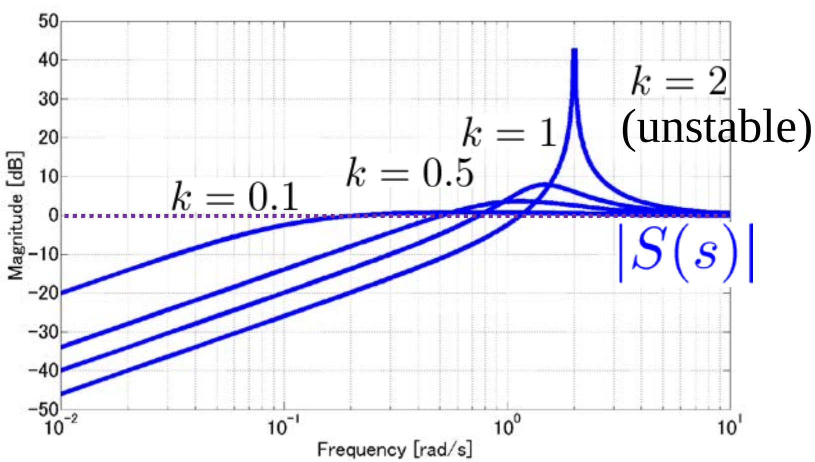
There exists a frequency range over which the magnitude of the sensitivity function exceeds 1 if it is to be kept below 1 at the other frequency range.



[SP05, Ex., p. 170]

$$P(s) = \frac{2-s}{2+s}, \quad K(s) = \frac{k}{s}$$

$$S(s) = \frac{1}{1+P(s)K(s)}$$



Maximum Peaks of $|S|$ and $|T|$ [SP05, p. 36]

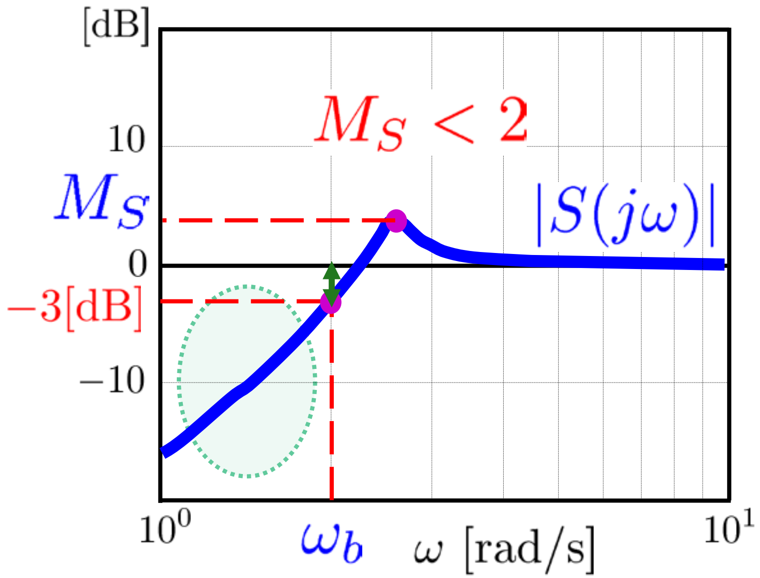
Sensitivity

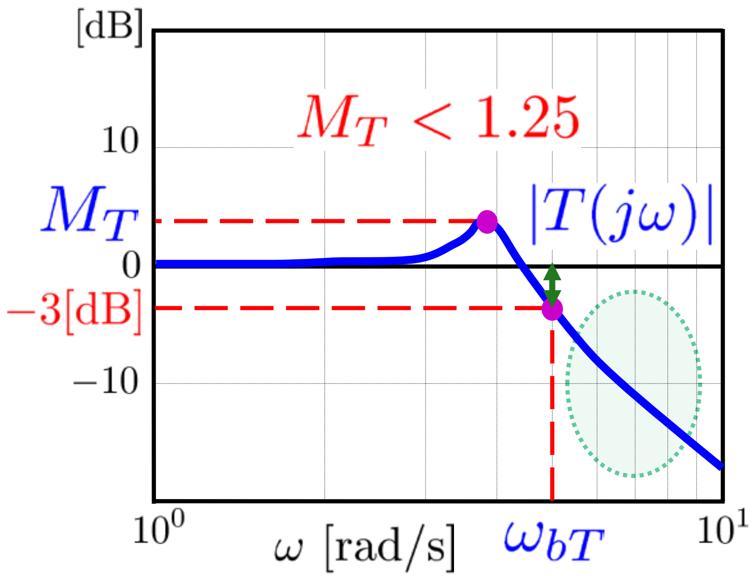
$$S = \frac{1}{1+PK}$$

Complementary Sensitivity

$$T = \frac{PK}{1+PK}$$

$$S + T = 1$$





M_S : Maximum Peak Magnitude of S

$$M_S = \max_{\omega} |S(j\omega)| < 2 \ (6 \text{ dB})$$

ω_b : Bandwidth Frequency of S

$$|S(j\omega_b)| = \frac{1}{\sqrt{2}} \ (-3 \text{ dB})$$

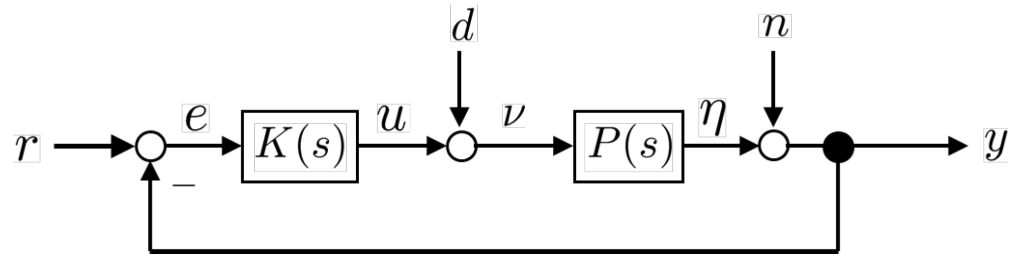
M_T : Maximum Peak Magnitude of T

$$M_T = \max_{\omega} |T(j\omega)| < 1.25 \ (2 \text{ dB})$$

ω_{bT} : Bandwidth Frequency of T

$$|T(j\omega_{bT})| = \frac{1}{\sqrt{2}} \ (-3 \text{ dB})$$

Loop Shaping

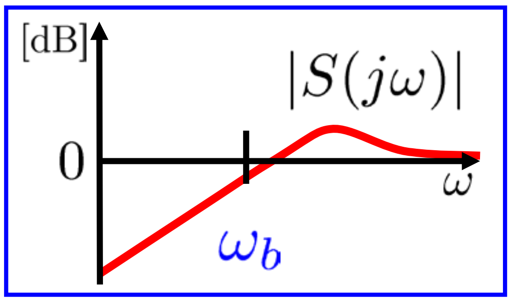


Loop Transfer Function

$$L(s) = P(s)K(s)$$

Sensitivity: $S = \dfrac{1}{1+L}$

Comp. Sensitivity: $T = \dfrac{L}{1+L}$



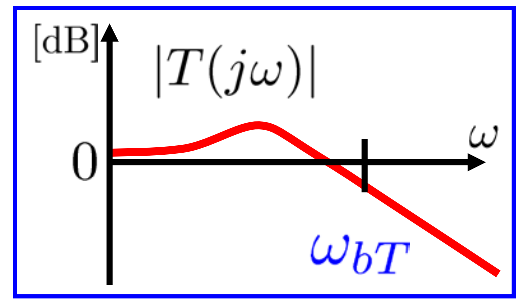
$|L| \gg 1 \rightarrow |S| \ll 1$

large small

+

Constraint

$S + T = 1$



$|L| \ll 1 \rightarrow |T| \ll 1$

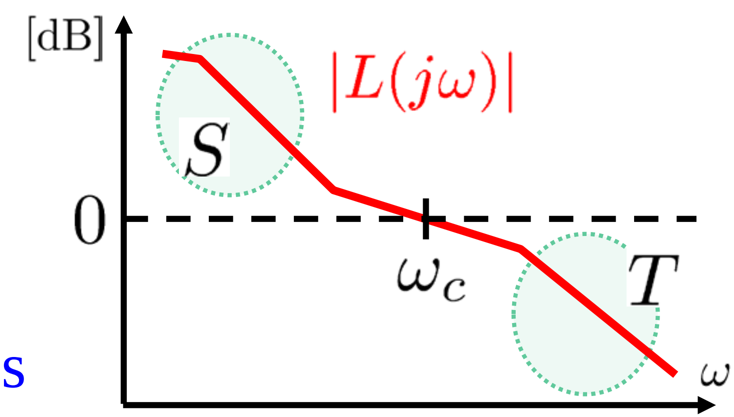
small small

Loop Shaping

Closed-loop S, T

➡ Open Loop L

Stability, Performance, Robustness



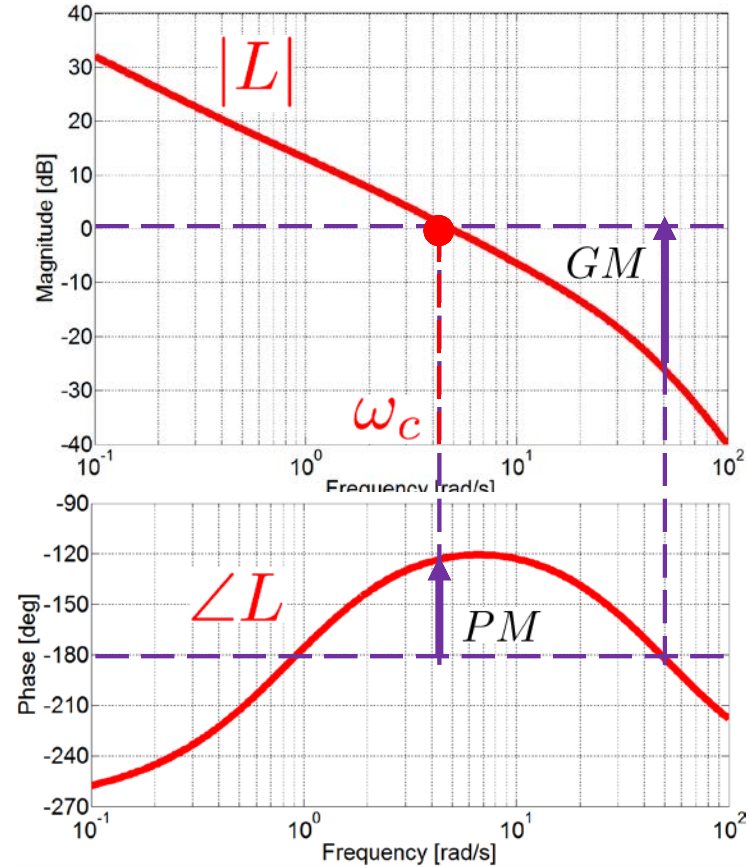
Loop Transfer Function

$$L(s) = P(s)K(s)$$

[SP05, Ex. 2.4] (p. 34)

$$P(s) = \frac{4}{(s-1)(0.02s+1)^2}$$

$$K(s) = 1.25\left(1 + \frac{1}{1.25s}\right)$$



Gain Crossover Frequency

$\omega_c = 4.9$ [rad/s] $\quad |L(j\omega_c)| = 1$

Stability Margins [SP05, p. 32]

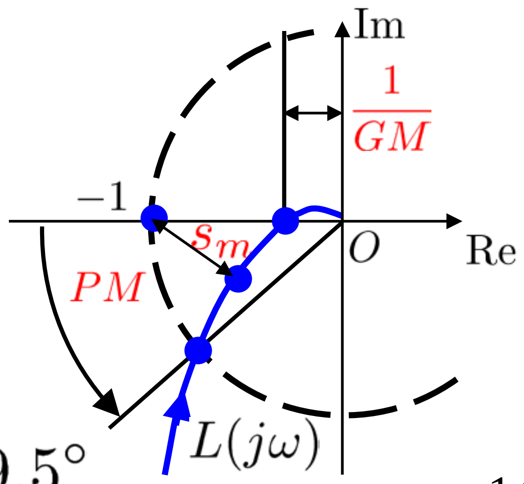
Gain Margin GM : $2 \sim 5$ ($6 \sim 14$ dB)

Phase Margin PM : $30^\circ \sim 60^\circ$

Time Delay Margin $\theta = PM/\omega_c$

Stability Margin $s_m = 1/M_S$: $0.5 \sim 0.8$

[SP05, Ex. 2.4] (p. 34) $GM = 18.7 \quad PM = 59.5^\circ$



Frequency Domain Performance

[SP05, Ex. 2.4] (p. 34)

$M_S = 1.19$ $M_T = 1.38$

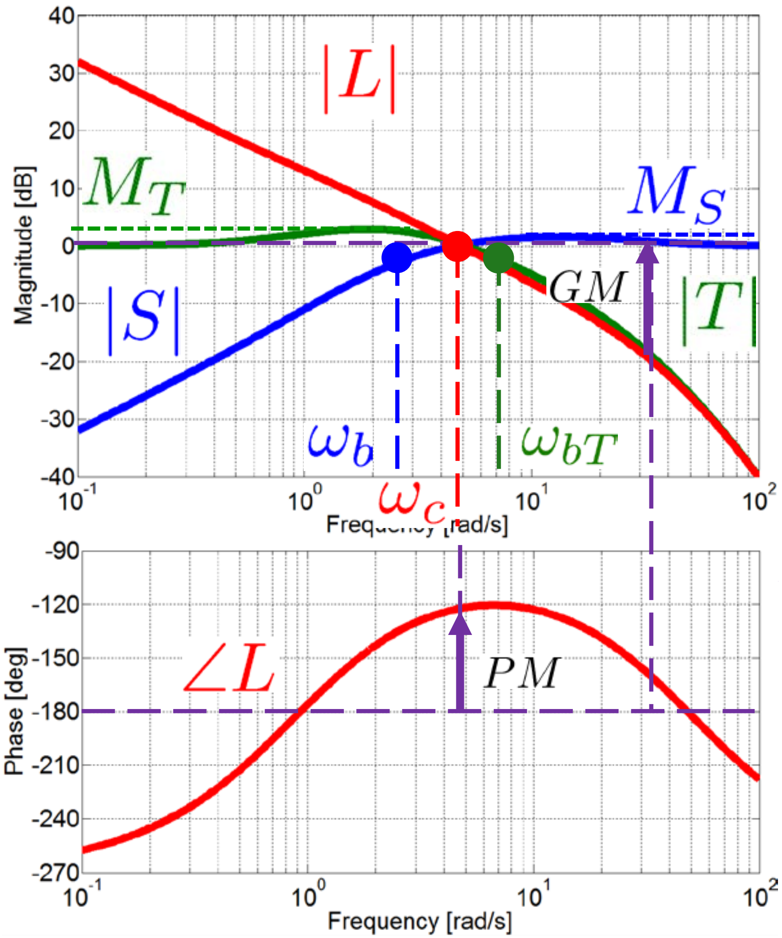
$M_S < 2$ $M_T < 1.25$

$\omega_b = 2.6$ [rad/s] $\omega_{bT} = 7.8$ [rad/s]

$\omega_c = 4.9$ [rad/s]

$\omega_b < \omega_c < \omega_{bT}$ $(PM < 90^\circ)$

$GM = 18.7$ $PM = 59.5^\circ$



Maximum Peak Criteria [SP05, p. 36]

$$GM \geq \frac{M_S}{M_S - 1}, PM \geq 2\sin^{-1}\left(\frac{1}{2M_S}\right) \text{ [rad]}$$

$$GM \geq 1 + \frac{1}{M_T}, PM \geq 2\sin^{-1}\left(\frac{1}{2M_T}\right) \text{ [rad]}$$

[Ex.] $M_S = 2$

$\rightarrow GM \geq 2,\ PM \geq 29.0^\circ$

[Ex.] $M_T = 1.25$

$\rightarrow GM \geq 1.8,\ PM \geq 46.0^\circ$

Bode Gain-phase Relationship [SP05, p. 18]

$$\angle G(j\omega_c) \approx \frac{\pi}{2}\underbrace{\left(\frac{d\ln|G(j\omega)|}{d\ln\omega}\right)_{\omega=\omega_c}}_{n_c}$$

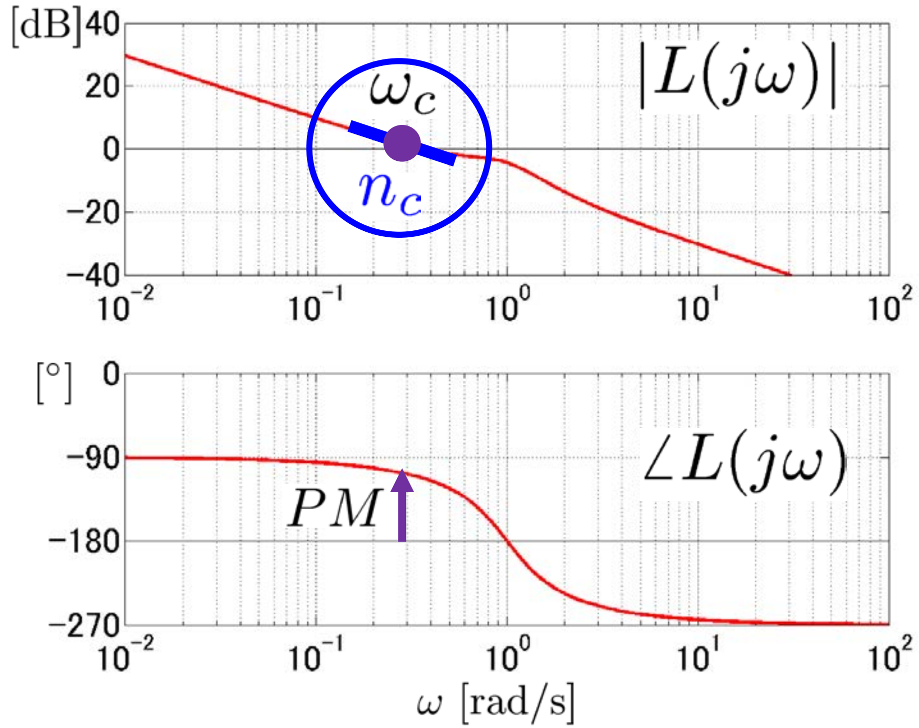
(minimum phase systems)

Slope of the Gain Curve at ω_c

$$n_c = -1 \rightarrow \angle G(j\omega_c) = -90^\circ$$

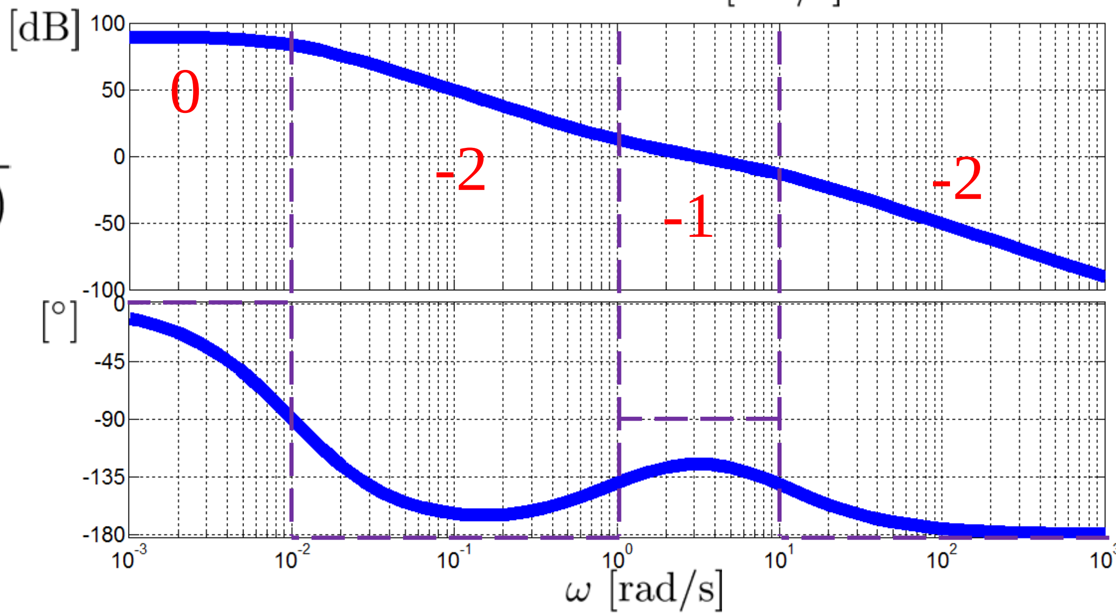
$$n_c = -2 \rightarrow \angle G(j\omega_c) = -180^\circ$$

Steep Slope: Small Phase Margin



[SP05, Ex., p. 20]

$$L(s) = \frac{30(s+1)}{(s+0.01)^2(s+10)}$$



Fundamental Limitations [SP05, pp. 183]

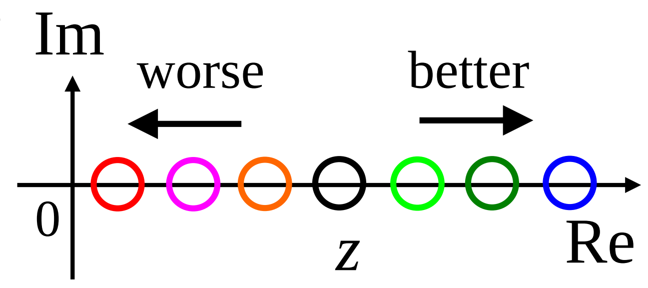
Bound on the Crossover Frequency ω_c

RHP (Right half-plane) Zero z $\quad \omega_c < \frac{z}{2}$

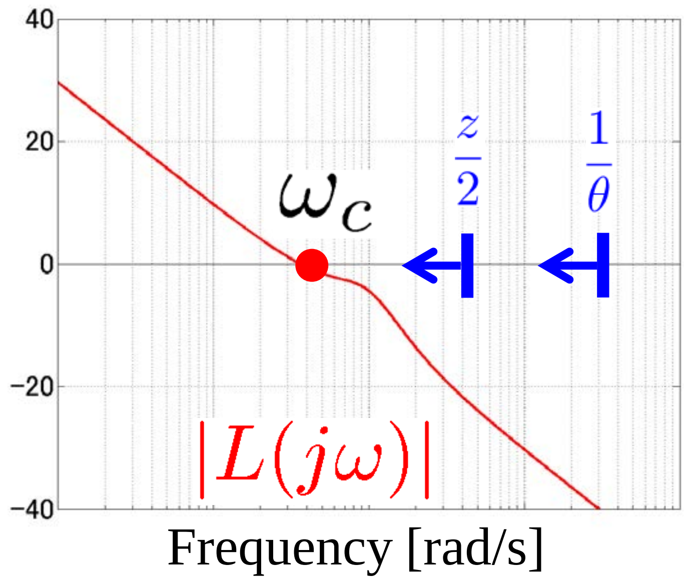
Fast RHP Zeros (z large): Loose Restrictions

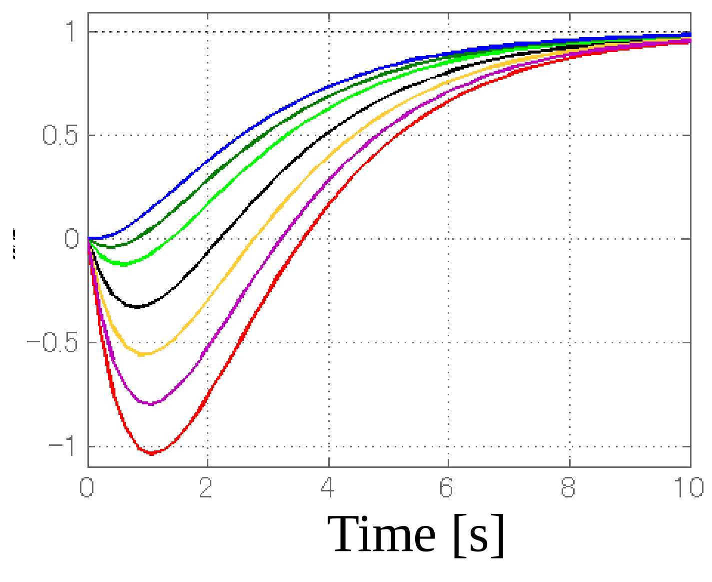
Slow RHP Zeros (z small): Tight Restrictions

Time Delay θ $\quad \omega_c < \frac{1}{\theta}$



Unstable zero





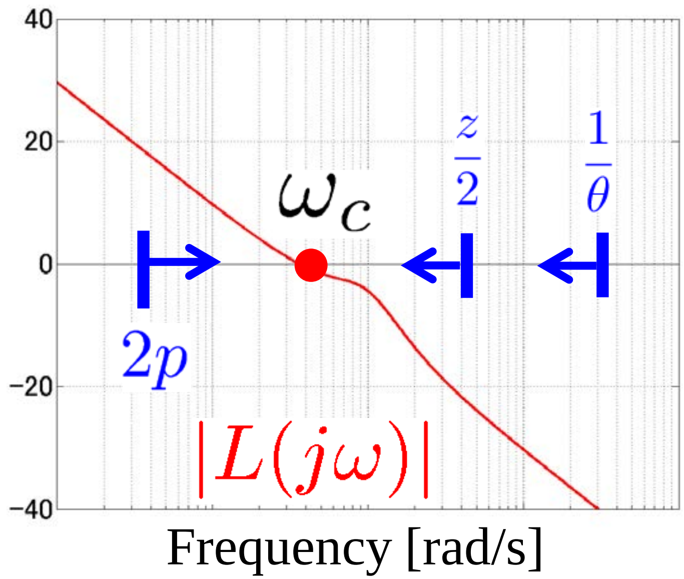
Fundamental Limitations [SP05, pp. 192, 194]

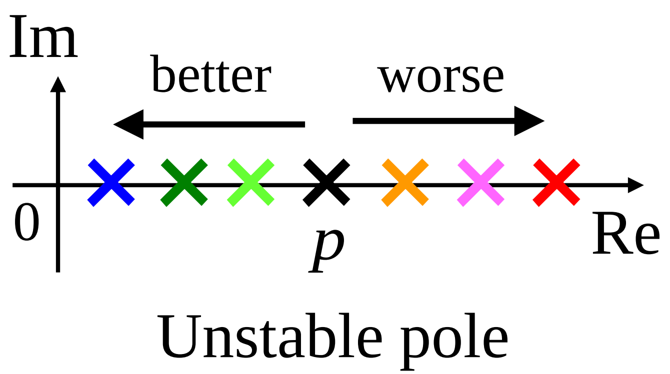
Bound on the Crossover Frequency ω_c

RHP (Right half-plane) Pole p $\omega_c > 2p$

Slow RHP Poles (p small): Loose Restrictions

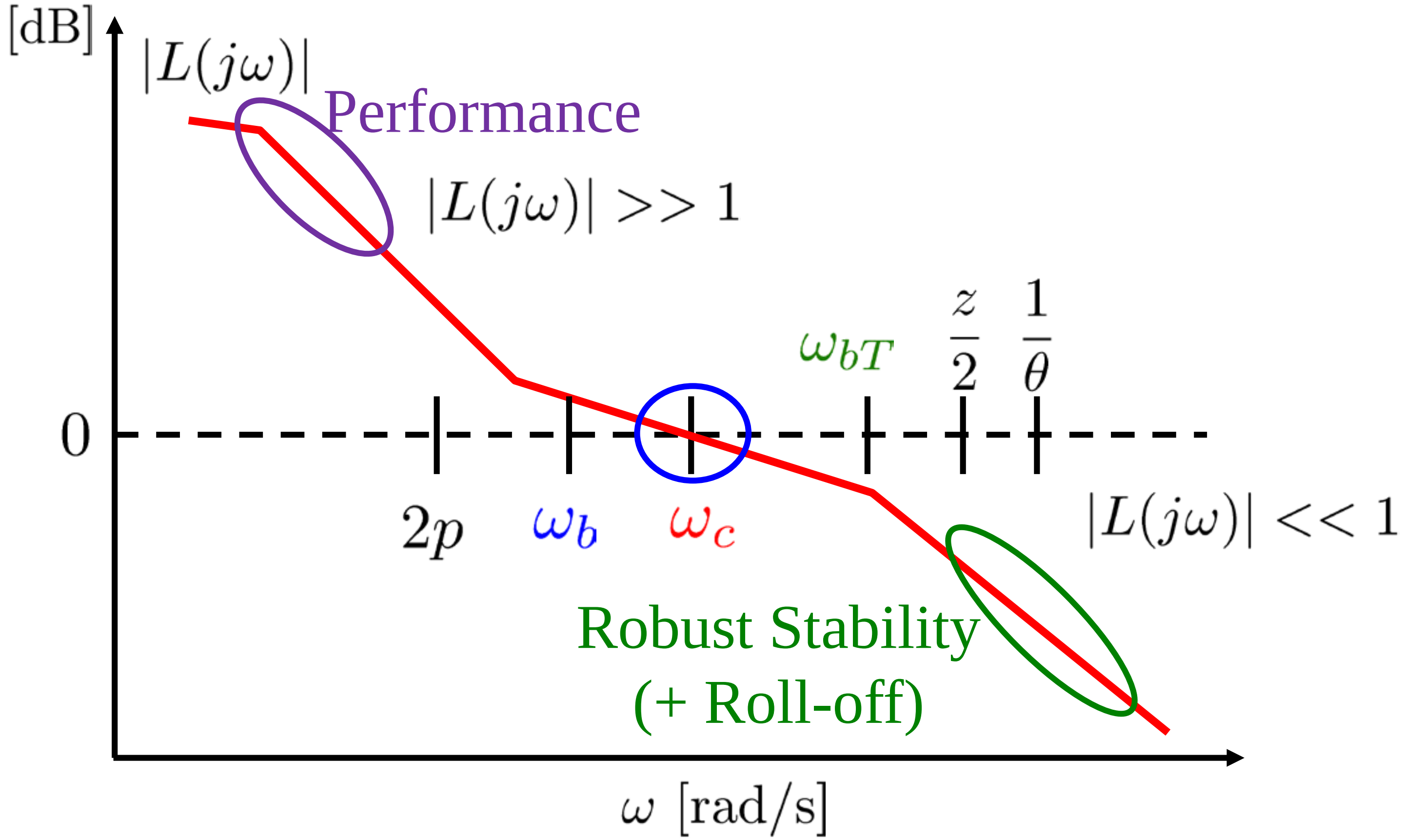
Fast RHP Poles (p large): Tight Restrictions





Poles on imaginary axis $\pm pj$ $\omega_c > 1.15|p|$

SISO Loop Shaping [SP05, pp. 41, 42, 343]

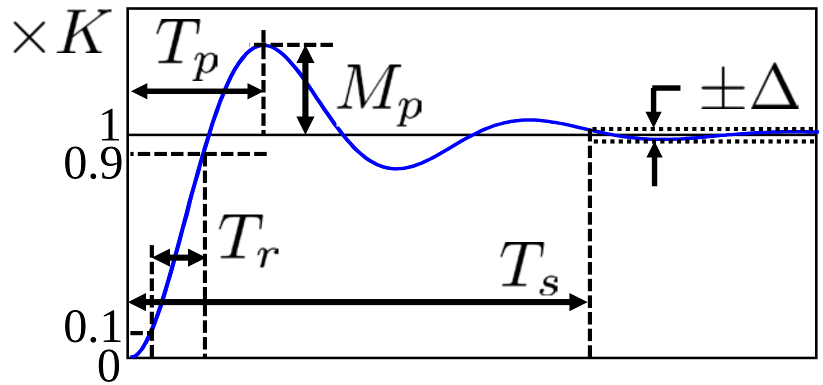


Loop Shaping Specifications

- Gain Crossover Frequency ω_c
- Shape of $L(j\omega)$
- System Type, Defined as the Number of Pure Integrators in $L(s)$
- Roll-off at Higher Frequencies

Step response analysis/Performance criteria

Rise time	T_r
Settling time	T_s
Peak time	T_p
Overshoot	M_p
Error tolerance	Δ



$K > 0$

First-order System

$$G_1(s) = \frac{K}{Ts+1} \quad T > 0$$

Rise time

$$T_r = (\ln 9)T \approx 2.2T$$

Settling time

$$T_s \approx \begin{cases} 3T & \text{if } \Delta = 5\% \\ 4T & \text{if } \Delta = 2\% \end{cases}$$

Overshoot

$$M_p = 0$$

Second-order System

$$G_2(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \omega_n > 0, \ \zeta \geq 0$$

Rise time $T_r = \frac{\pi/2 + \arcsin\zeta}{\omega_n\sqrt{1-\zeta^2}}$

Settling time

$$T_s \approx \begin{cases} 3/\zeta\omega_n & \text{if } \Delta = 5\% \\ 4/\zeta\omega_n & \text{if } \Delta = 2\% \end{cases}$$

Overshoot $M_p = Ke^{-\zeta\pi/\sqrt{1-\zeta^2}}$

Peak Time $T_p = \pi/(\omega_n\sqrt{1-\zeta^2})$

[QZ07] L. Qiu and K. Zhou (2007) *Introduction to Feedback Control*, Prentice Hall.

Design Relations

Maximum Peak Magnitude of T

$$M_T \simeq \frac{1}{2\sin(PM/2)}$$

Phase Margin

$$PM = \tan^{-1}\left[\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4}-2\zeta^2}}\right]$$

$$\approx 100\zeta \quad (< 70^\circ)$$

Bandwidth

$$\omega_c \le \omega_{bT} \le 2\omega_c$$

$$\omega_{bT} = \omega_c \quad \text{if} \quad PM = 90^\circ$$

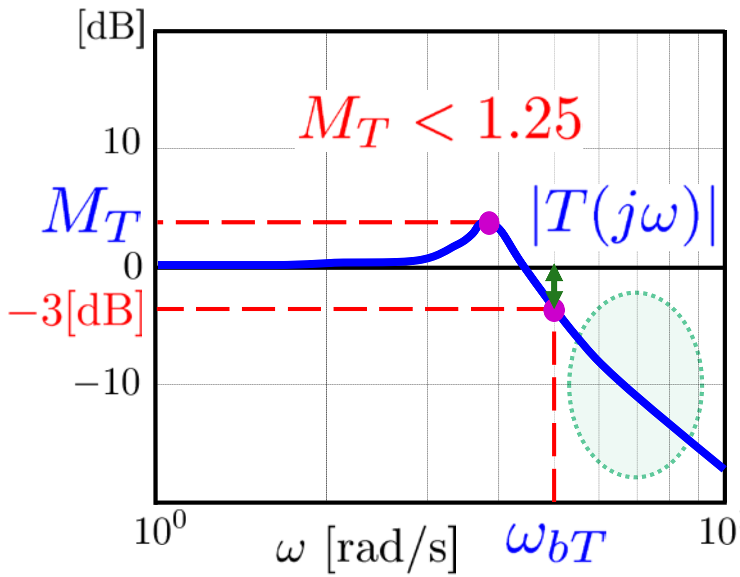
$$\omega_{bT} \simeq 2\omega_c \quad \text{if} \quad PM \le 45^\circ$$

$$0 < \omega_b \le \omega_c$$

$$\omega_b = \omega_c \quad \text{if} \quad PM = 90^\circ$$

Complementary Sensitivity

$$T = \frac{PK}{1+PK}$$



M_T: Maximum Peak Magnitude of T

$$M_T = \max_{\omega} |T(j\omega)| < 1.25 \ (2 \text{ dB})$$

ω_{bT}: Bandwidth Frequency of T

$$|T(j\omega_{bT})| = \frac{1}{\sqrt{2}} \quad (-3 \text{ dB})$$

[FPN09] G.F. Franklin, J.D. Powell and A. E.-Naeini (2009) *Feedback Control of Dynamic Systems*, Sixth Edition, Prentice Hall.

Controllability analysis with SISO feedback control

[SP05, pp. 206-209]

M_1 Margin to stay within constraints $|u| < 1$

M_2 Margin for performance $|e| < 1$

M_3 Margin because of RHP-pole p

$$2p < \omega_c$$

M_4 Margin because of RHP-zero z

$$\omega_c < z/2$$

M_5 Margin because of frequency ω_u where plant has -180° phase lag

$$\omega_c < \omega_u$$

$\omega_d <$

M_6 Margin because of delay θ

ω_c

$$\omega_c < 1/\theta$$

Typically, the closed-loop bandwidth of the spacecraft is an order of magnitude less than the lowest mode frequency, and as long as the controller does not excite any of the flexible modes, the sampling period may be selected solely based on the closed-loop bandwidth.

[Le10] W.S. Levine (Eds.) (2010) *The Control Handbook, Second Edition: Control System Fundamentals, Second Edition*, CRC Press.

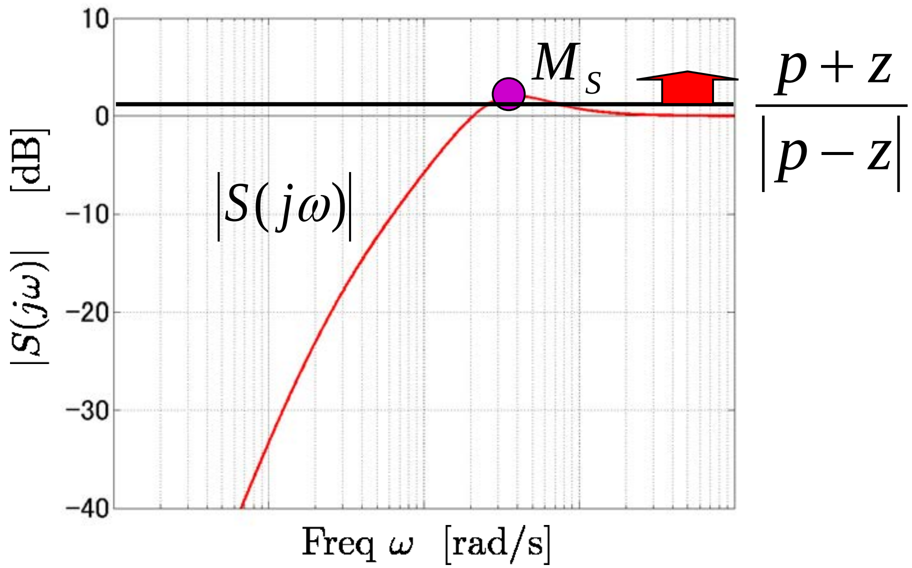
RHP Poles/Zeros, Time Delays and Sensitivity

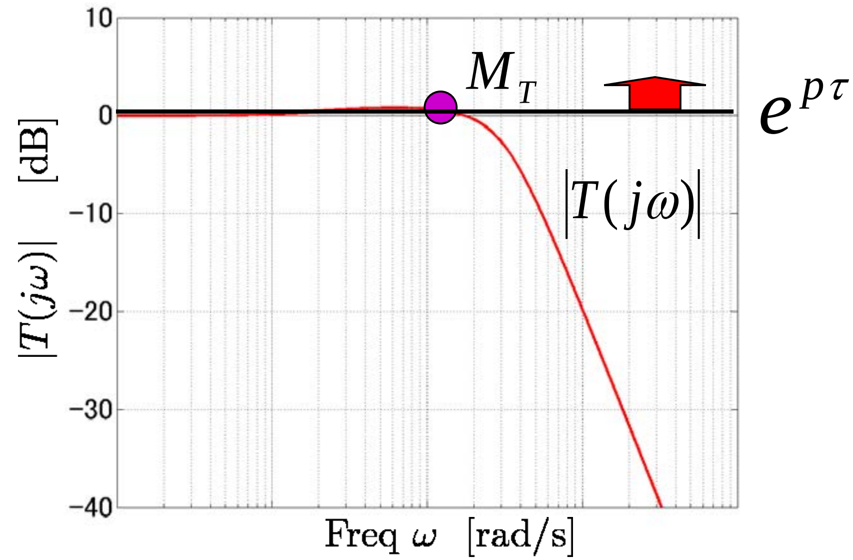
For systems with a RHP pole p and RHP zero z (or a time delay τ), any stabilizing controller gives sensitivity functions with the property

$$M_S = \sup_{\omega} |S(j\omega)| \geq \frac{p+z}{|p-z|}$$

$$M_T = \sup_{\omega} |T(j\omega)| \geq e^{p\tau}$$

RHP pole and zero and time delay significantly limit the achievable performance of a system

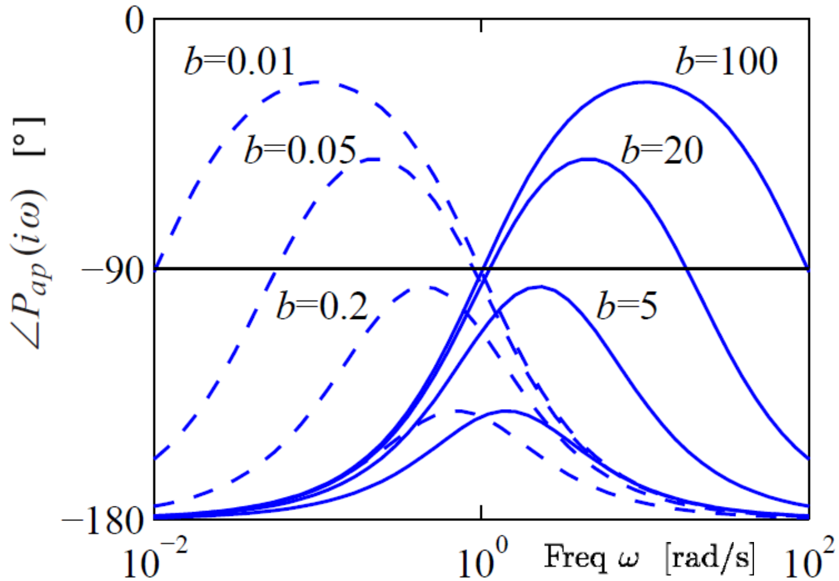




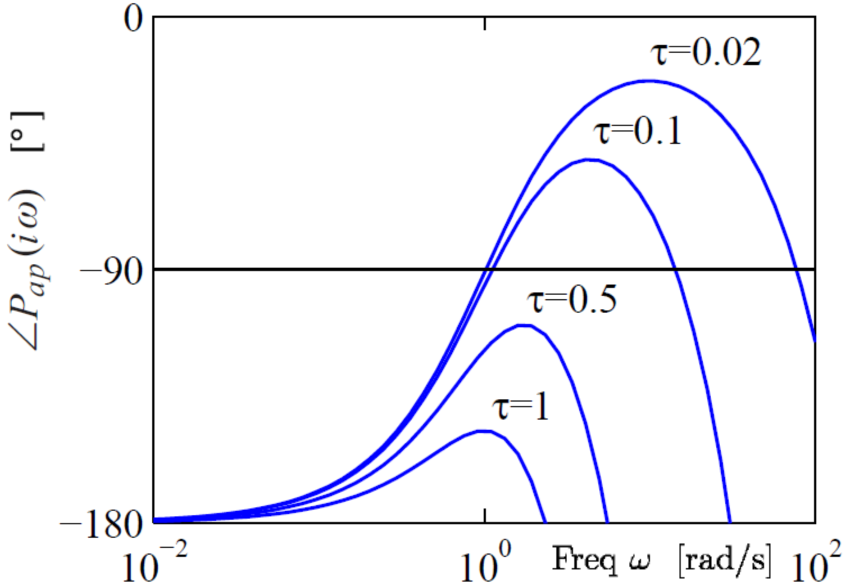
RHP Poles/Zeros, Time Delays and Sensitivity

All-pass system($p = 1$, $z = b$, τ)

$$P_{ap}(s) = \frac{b - s}{s - 1} \qquad P_{ap}(s) = \frac{e^{-s\tau}}{s - 1}$$



RHP pole/zero pair



RHP pole and time delay

$z / p < 1/6$ or $6 < z / p$

$p\tau < 0.3$

The zero and the pole must be sufficiently far apart

The product of RHP pole and time delay must be sufficiently small

allowable phase lag of P_{ap} at ω_{gc}: $\varphi_l = 90^\circ$