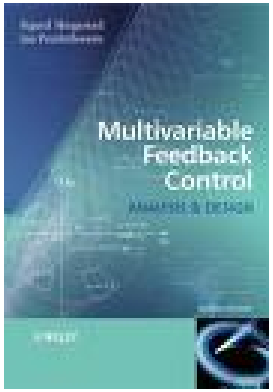
Robust and Optimal Control, Spring 2015

Instructor: Prof. Masayuki Fujita (S5-303B)

C. H_∞ Loop Shaping Design

Reference:

[SP05] S. Skogestad and I. Postlethwaite,
Multivariable Feedback Control; Analysis and Design,
Second Edition, Wiley, 2005.



Robust Stabilization

K. Glover and D. McFarlane, "Robust stabilization of normalized coprime factor plant descriptions with H_∞ bounded uncertainty," IEEE TAC, **34**-8, 821-830, 1989

D. McFarlane K. Glover

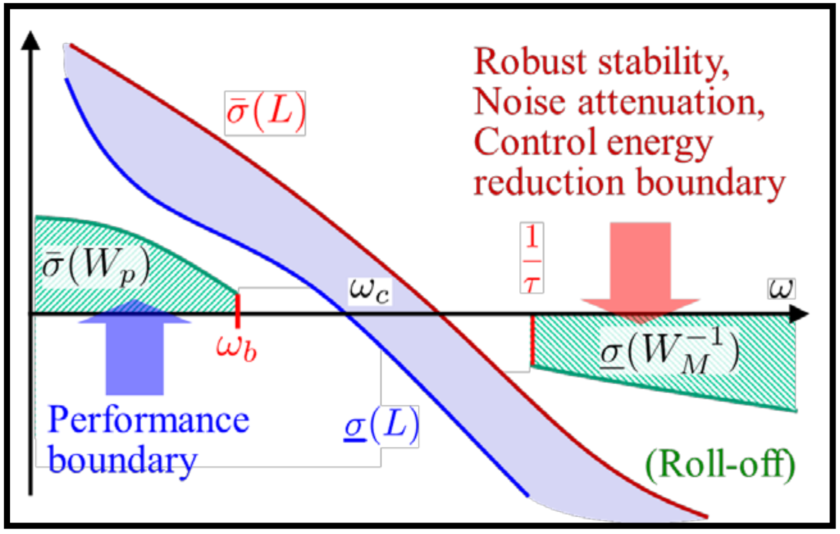
Mixed Sensitivity Problem

$$\text{find } K(s) \text{ s.t. } \left\| \begin{matrix} W_P(s)S(s) \\ W_M(s)T(s) \end{matrix} \right\|_\infty < 1$$

Remark

Pole/Zero Cancellation

Loop Shaping (NS, NP+RS)



Perturbations to Coprime Factors

Coprime Factorization of Transfer Functions

$p \times m$ Transfer Function Matrix

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] = M_l^{-1} N_l = N_r M_r^{-1}$$

(Left) (Right)

(A, B) Controllable (A, C) Observable

$$M_l, N_l, M_r, N_r \in \mathcal{H}_\infty$$

$$\text{rank} \begin{bmatrix} N_l & M_l \end{bmatrix} = p, \ \forall \text{Re}(s) \geq 0 \quad \text{rank} \begin{bmatrix} N_r \\ M_r \end{bmatrix} = m, \ \forall \text{Re}(s) \geq 0$$

[There are no “common zeros” in N and M in the right half plane]

[Ex.] $G(s) = N(s) M^{-1}(s) = \dfrac{1}{s}$

$$\Rightarrow \quad N(s) = \frac{1}{s+1}, \quad M^{-1}(s) = \frac{s+1}{s}$$

Normalized Coprime Factorization

(NLCF: Normalized Left-Coprime Factorization)

$$M_l(j\omega)M_l(j\omega)^* + N_l(j\omega)N_l(j\omega)^* = I, \ \forall\omega$$
$$\left[\ M(s)^* = M^T(-s)\ \right]$$

Note: Given any coprime factorization of $G = M_l^{-1}N_l$ then for R (need the poles and zeros of R to be in the LHP)

$$G = (RM_l)^{-1}(RN_l)$$
$$(RM_l)(M_l^* R^*) + (RN_l)(N_l^* R^*) = R(M_l M_l^* + N_l N_l^*)R^*$$

[Ex.] $G(s) = \dfrac{N(s)}{M(s)} = \dfrac{1}{s}$ ➡ $N(s) = \dfrac{1}{s+1}$, $M(s) = \dfrac{s}{s+1}$

$$MM^* + NN^* = \frac{j\omega}{j\omega+1}\cdot\frac{-j\omega}{-j\omega+1} + \frac{1}{j\omega+1}\cdot\frac{1}{-j\omega+1}$$
$$= \frac{\omega^2}{1+\omega^2} + \frac{1}{1+\omega^2} = 1$$

Coprime Factor Uncertainty [SP05, pp. 304, 365]

Nominal Plant Model [SP05, p. 122]

(Left Coprime Factorization)

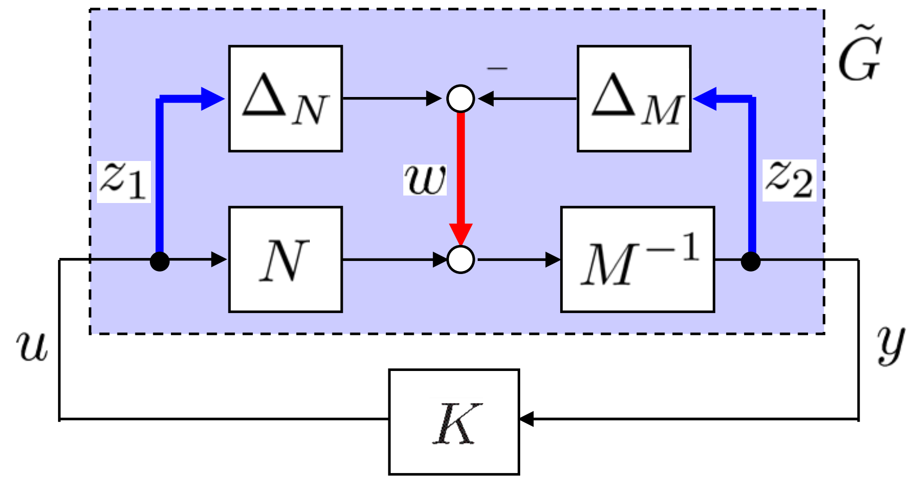
$$G = M^{-1}N$$

The set of Plant Models

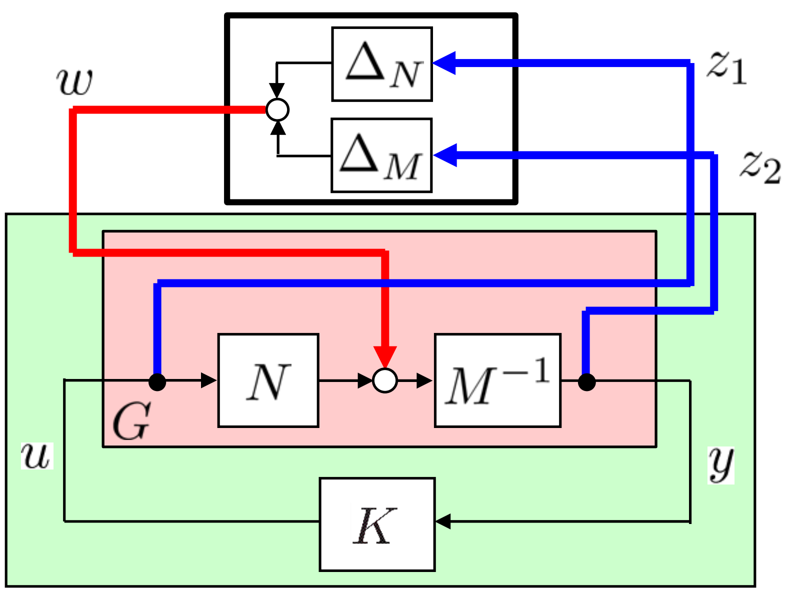
$$\tilde{G} = (M + \Delta_M)^{-1}(N + \Delta_N)$$

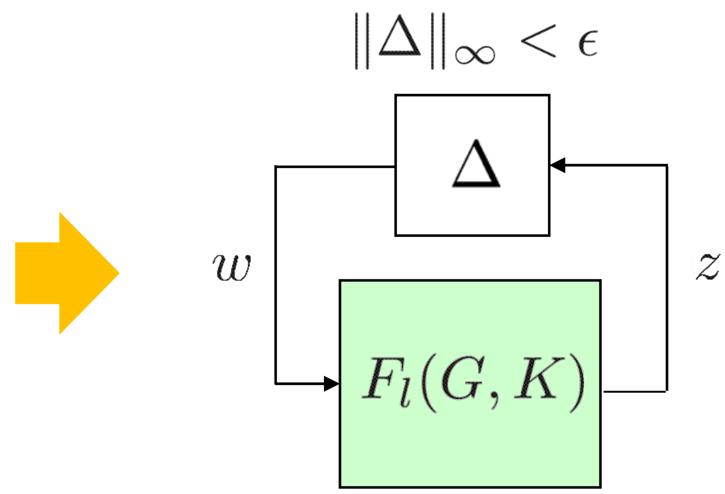
$$\left\| \begin{bmatrix} \Delta_N & \Delta_M \end{bmatrix} \right\|_\infty < \epsilon$$

$$M, N, \Delta_M, \Delta_N \in \mathcal{RH}_\infty$$



Full-matrix



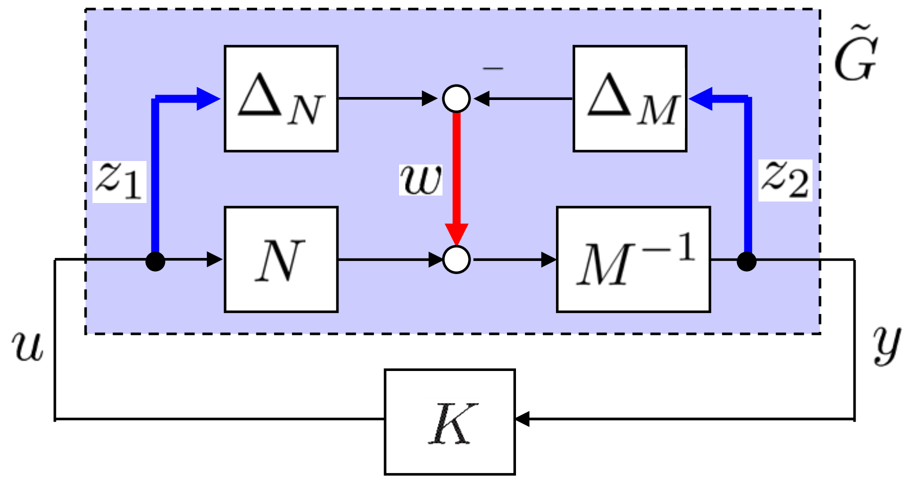


Coprime Factor Uncertainty

Nominal Plant Model [SP05, p. 122]

(Left Coprime Factorization)

$$G = M^{-1}N$$



The set of Plant Models

$$\tilde{G} = (M + \Delta_M)^{-1}(N + \Delta_N)$$

$$\| [\ \Delta_N \quad \Delta_M\ ] \|_\infty < \epsilon$$

$$M, N, \Delta_M, \Delta_N \in \mathcal{RH}_\infty$$

[Ex.] $G(s) = \dfrac{1}{s}$ ➡ $N(s) = \dfrac{1}{s+1}$, $M(s) = \dfrac{s}{s+1}$

$$\tilde{G}(s) = \frac{N + \Delta_N}{M + \Delta_M} = \frac{1 + \Delta_N(s+1)}{s + \Delta_M(s+1)}$$ with $|\Delta_N|^2 + |\Delta_M|^2 < \epsilon^2$

If Δ_M is real constant, then pole is moved to $-\dfrac{\Delta_M}{1+\Delta_M}$. Hence poles move across $s = j\omega$ with small $|\Delta_M|$ but very large $|G_\Delta - G_o|$ changes.

Note: Coprime factor perturbations are not unique.

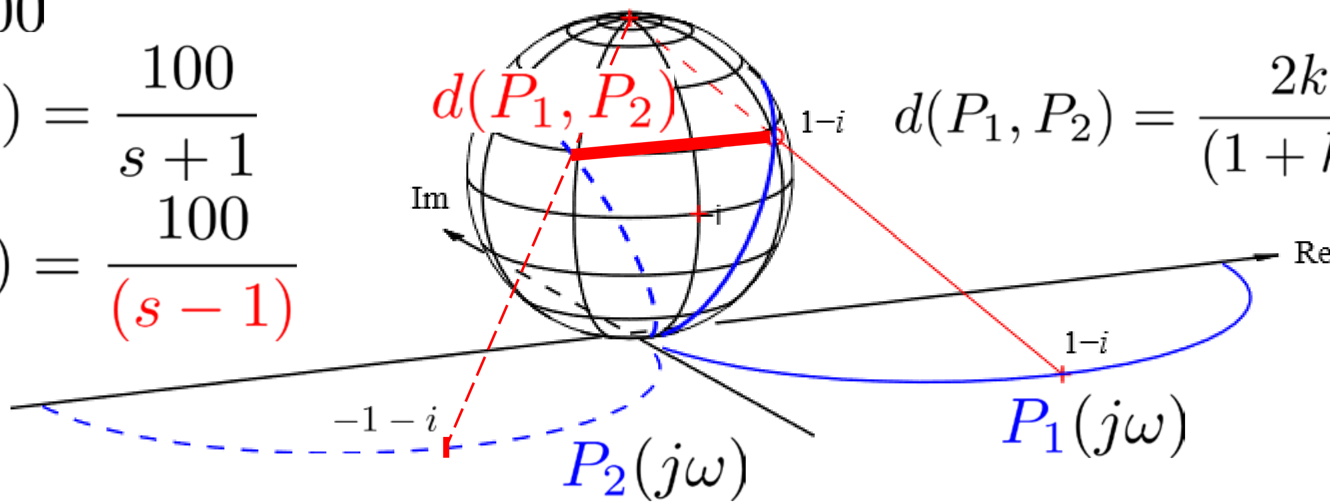
Vinnicombe Metric (ν-gap Metric)

[ZD97, Chap.17]
[AM09, pp. 349-352]

$k = 100$

$$P_1(s) = \frac{100}{s+1}$$

$$P_2(s) = \frac{100}{(s-1)}$$



$$d(P_1, P_2) = \frac{2k}{(1+k^2)} = \frac{200}{10001}$$

Vinnicombe metric (ν-gap Metric)

$$\delta_v(P_1, P_2) = d(P_1, P_2) \in [0, 1] \quad \text{if } (P_1, P_2) \in \mathcal{C}$$

G. Vinnicombe

A distance measure that is appropriate for closed loop systems

$$d(P_1, P_2) = \sup_{\omega} \frac{|P_1(j\omega) - P_2(j\omega)|}{\sqrt{(1 + |P_1(j\omega)|^2)(1 + |P_2(j\omega)|^2)}} \in [0, 1]$$

[AP09, Ex 12.2] $\delta_v(P_1, P_2) = 0.98$

[AP09, Ex 12.3] $\delta_v(P_1, P_2) = 0.02$

[ZD97] K. Zhou with J.C. Doyle, *Essentials of Robust Control*, Prentice Hall, 1997.

Robust Stability Condition [SP05, pp. 305, 366]

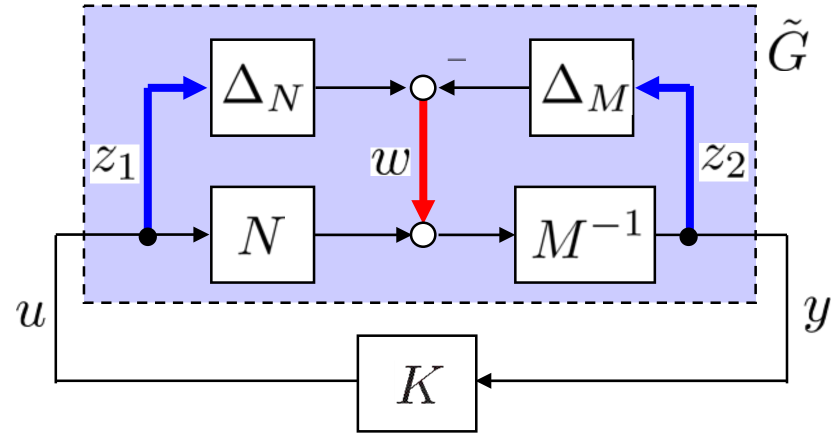
Plant Model

$$G = M^{-1}N$$

The set of Plant Models

$$\tilde{G} = (M + \Delta_M)^{-1}(N + \Delta_N)$$

$$\left\| \begin{bmatrix} \Delta_N & \Delta_M \end{bmatrix} \right\|_\infty < \epsilon$$



Closed loop system (LFT)

$$\|F_l(G,K)\|_\infty = \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} M^{-1} \right\|_\infty$$

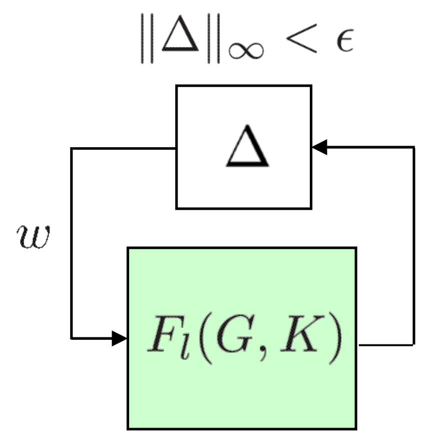
$$= \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} M^{-1} \begin{bmatrix} M & N \end{bmatrix} \right\|_\infty$$

$$= \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} \begin{bmatrix} I & G \end{bmatrix} \right\|_\infty$$

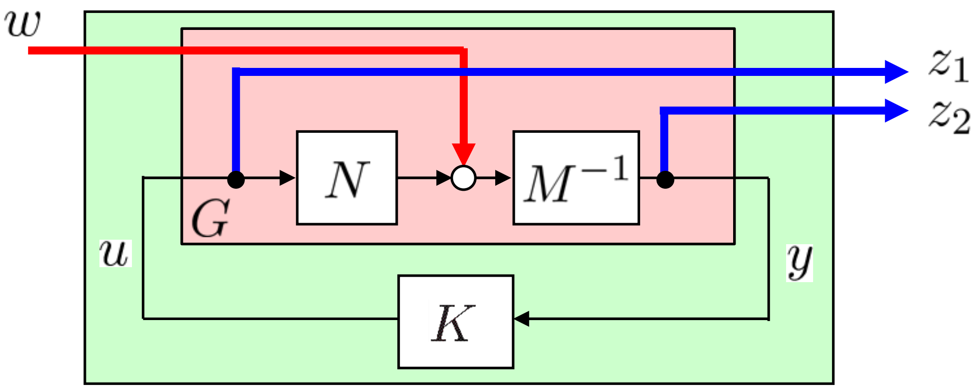
Robust Stability Condition

$$\gamma = \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} \begin{bmatrix} I & G \end{bmatrix} \right\|_\infty \le \frac{1}{\epsilon}$$

($\because$ Small Gain Theorem)



H_∞ Control Synthesis



Nominal Plant Model

$$G = M_l^{-1} N_l = N_r M_r^{-1}$$

Double Bezout Equation

$$\begin{bmatrix} V_l & -U_l \\ -N_l & M_l \end{bmatrix} \begin{bmatrix} M_r & U_r \\ N_r & V_r \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

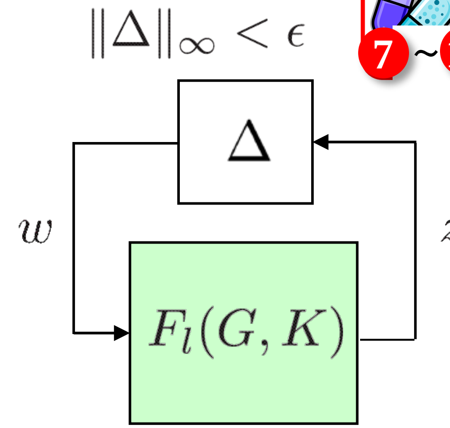
All Stabilizing Controllers (Youla Parameterization)

$$\begin{aligned} K &= (V_l + Q N_l)^{-1} (U_l + Q M_l) \\ &= (U_r + M_r Q)(V_r + N_r Q)^{-1} \quad \text{for} \quad Q \in \mathcal{H}_\infty \end{aligned}$$

Closed-loop Transfer Function

$$\begin{aligned} F_l(G, K) &= \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} \begin{bmatrix} I & G \end{bmatrix} \\ &= \begin{bmatrix} U_r \\ V_r \end{bmatrix} \begin{bmatrix} M_l & N_l \end{bmatrix} + \begin{bmatrix} M_r \\ N_r \end{bmatrix} Q \begin{bmatrix} M_l & N_l \end{bmatrix} \end{aligned}$$

NLCF Robust Control Problem [SP05, p. 366]



H_∞ Sub-optimal Control Problem

Given $\gamma > \gamma_{\min}$, find all stabilizing controllers K such that $\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} \begin{bmatrix} I & G \end{bmatrix} \right\|_\infty \leq \gamma = \frac{1}{\epsilon}$

Solution

Minimum Value of H_∞-norm (Maximum Stability Margin)

$$\gamma_{\min}^2 = \epsilon_{\max}^{-2} = 1 + \underline{\lambda_{\max}(XZ)}$$

Maximum eigenvalues of the matrix XZ

Sub-optimal Solution (H_∞ controller)

$$K = (K_{11}\Phi + K_{12})(K_{21}\Phi + K_{22})^{-1}$$

Φ : a transfer function satisfying $\|\Phi\|_\infty < 1$

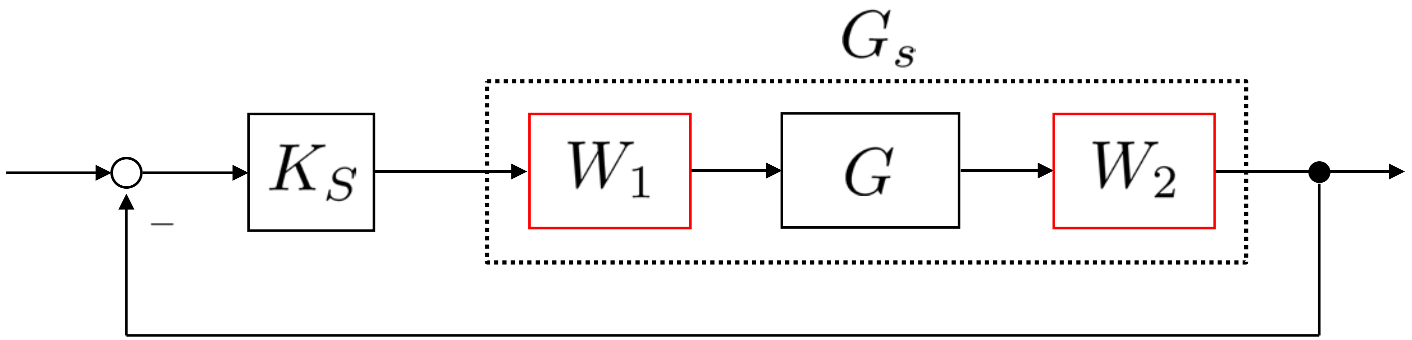
$$\gamma^{-1} = \epsilon_{\max} = \sqrt{1 - \|[\tilde{N}_S \;\; \tilde{M}_S]\|_H^2} \leq 1$$

Central controller ($\Phi = 0$)

$$K = \left[\begin{array}{c|c} A + BF + \gamma^2 W_1^{-T} Z C^T (C + DF) & \gamma^2 W_1^{-T} X C^T \\ \hline B^T X & -D^T \end{array} \right]$$

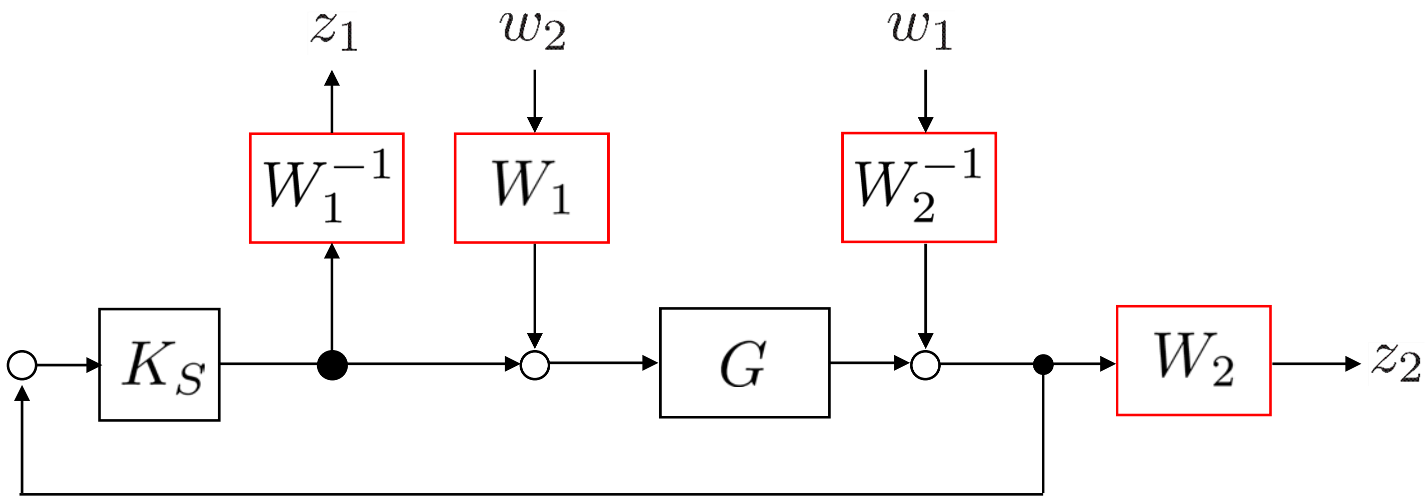
Loop Shaping Design Procedure (LSDP)

D. McFarlane and K. Glover, "A loop Shaping Design Procedure Using H_∞ Synthesis," IEEE TAC, **37**-6, 759-769, 1992



The shaped plant and controller

Loop Shaping Design

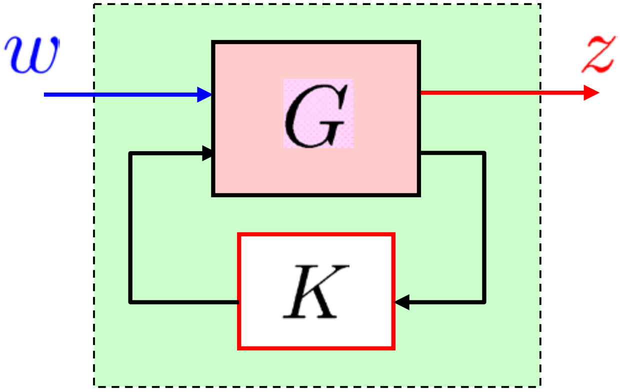


$$F_l(G,K)$$

$$= \begin{bmatrix} W_1^{-1}K \\ W_2 \end{bmatrix} (I - GK)^{-1} \begin{bmatrix} W_2^{-1} & GW_1 \end{bmatrix}$$

$$= \begin{bmatrix} K_s \\ I \end{bmatrix} (I - G_S K_S)^{-1} \begin{bmatrix} I & G_S \end{bmatrix}$$

$$G_S = W_2 G W_1, \; K = W_1 K_S W_2$$



$$z = F_l(G,K)w$$

Loop Shaping Design Procedure (LSDP) [SP05, p. 368]

STEP 1: Loop Shaping $G_s = W_2 G W_1$

To shape the plant G using shaping functions W_1 and W_2

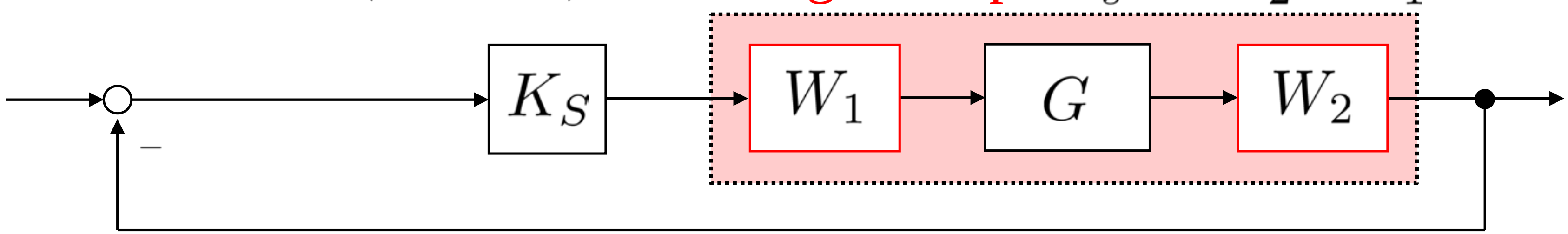
W_1, W_2 are chosen to satisfy keeping unstable pole of the model G

STEP 2: Robust Stabilization

Find K_S such that $\left\| \begin{bmatrix} K_s \\ I \end{bmatrix} (I - G_S K_S)^{-1} \begin{bmatrix} I & G_S \end{bmatrix} \right\|_\infty \leq \gamma$

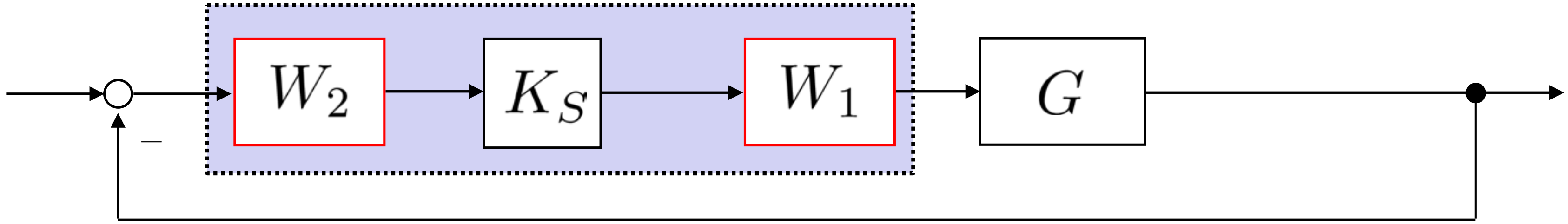
(maximize $b_0(G_S, K_S)$)

Target Loop $G_s = W_2 G W_1$

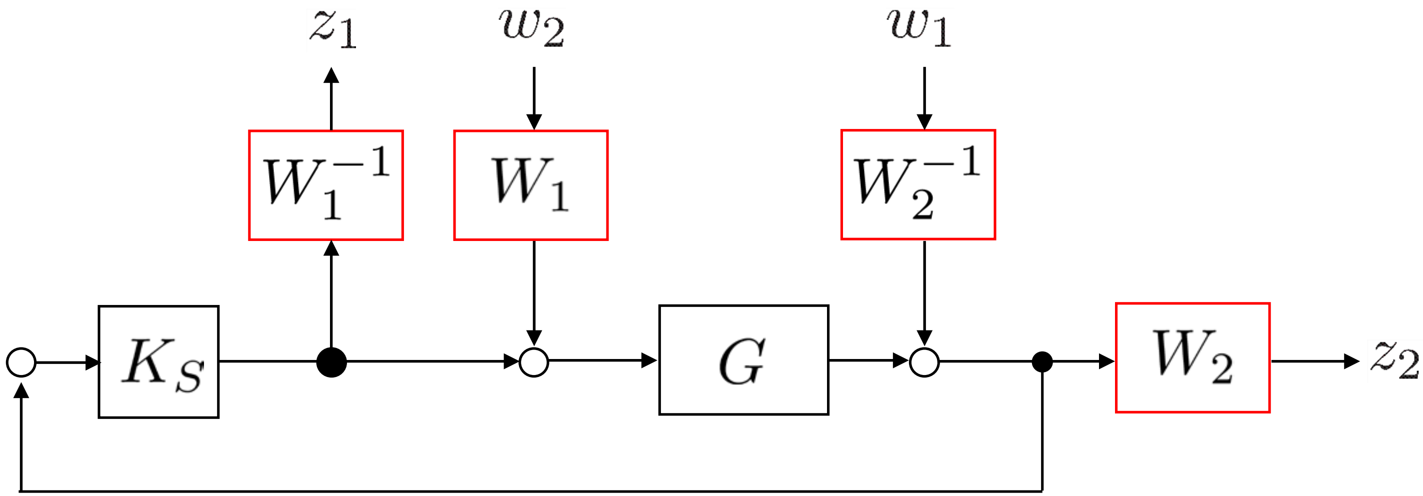


STEP 3: H_∞ Controller $K_\infty = W_1 K_S W_2$

K_∞



LSDP: Decision Method of Weights W_1, W_2



$$G_s = W_2 G W_1 = \tilde{M}_s^{-1} \tilde{N}_s$$

$$\bar{\sigma}(\tilde{N}_s) = \left(\frac{\bar{\sigma}^2(G_s)}{1 + \bar{\sigma}^2(G_s)} \right)^{1/2} \qquad \bar{\sigma}(\tilde{M}_s) = \left(\frac{1}{1 + \underline{\sigma}^2(G_s)} \right)^{1/2}$$

$$\begin{cases} \bar{\sigma}(K(I - GK)^{-1}) \leq \gamma \bar{\sigma}(\tilde{M}_s) \bar{\sigma}(W_1) \bar{\sigma}(W_2) \\ \bar{\sigma}((I - GK)^{-1}) \leq \gamma \bar{\sigma}(\tilde{M}_s) c(W_2) \\ \bar{\sigma}(K(I - GK)^{-1} G) \leq \gamma \bar{\sigma}(\tilde{N}_s) c(W_1) \\ \bar{\sigma}((I - GK)^{-1} G) \leq \dfrac{\gamma \bar{\sigma}(\tilde{N}_s)}{\underline{\sigma}(W_1) \underline{\sigma}(W_2)} \end{cases}$$

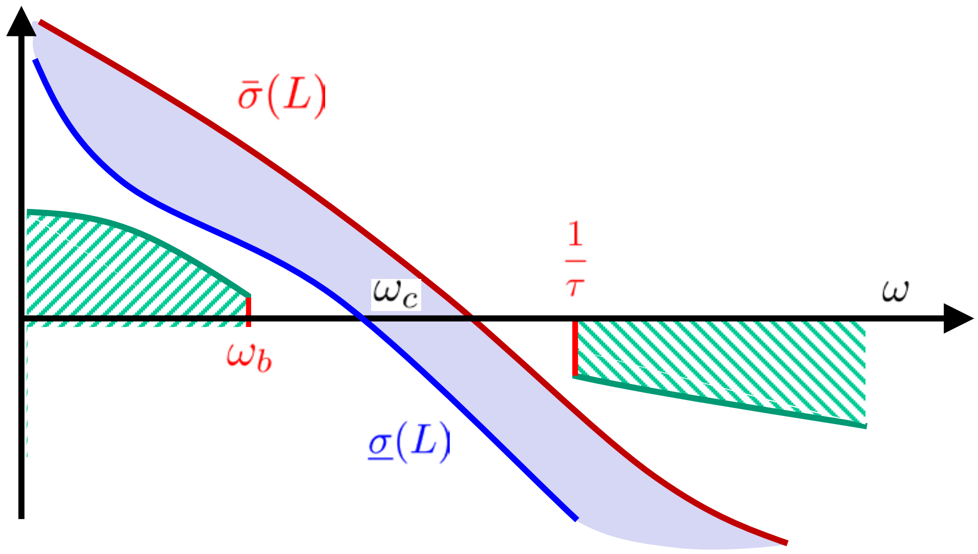
LSDP in MIMO Systems

For Low Frequencies $\underline{\sigma}(W_2 G W_1) \gg 1$

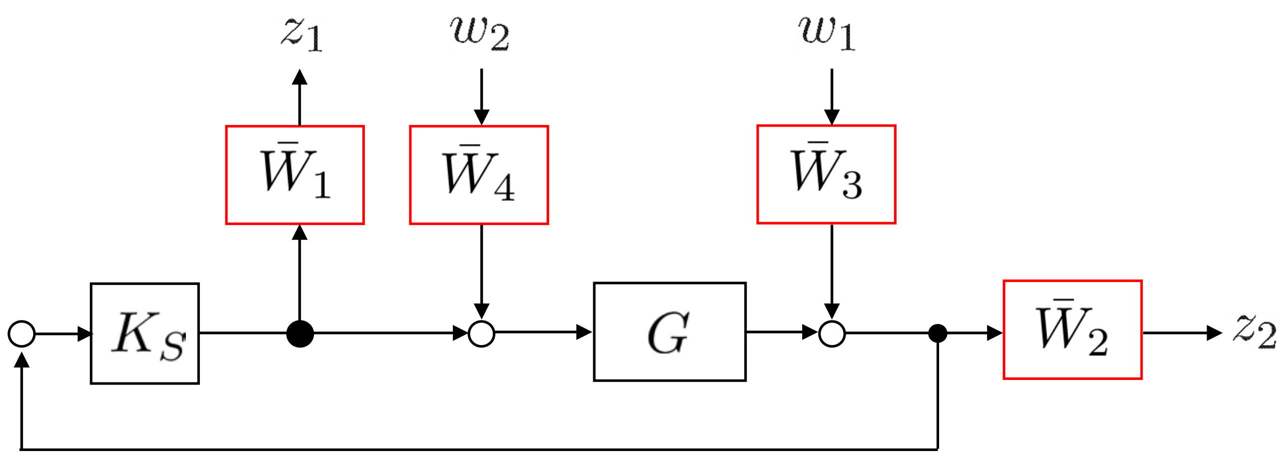
$$\Rightarrow \begin{cases} \bar{\sigma}((I-GK)^{-1}) \le \dfrac{\gamma}{\underline{\sigma}(G)\underline{\sigma}(W_1)\underline{\sigma}(W_2)} \\ \bar{\sigma}((I-GK)^{-1}G) \le \dfrac{\gamma}{\underline{\sigma}(W_1)\underline{\sigma}(W_2)} \end{cases}$$

For High Frequencies $\bar{\sigma}(W_2 G W_1) \ll 1$

$$\Rightarrow \begin{cases} \bar{\sigma}(K(I-GK)^{-1}) \le \gamma\bar{\sigma}(W_1)\bar{\sigma}(W_2) \\ \bar{\sigma}(K(I-GK)^{-1}G) \le \gamma\bar{\sigma}(G)\bar{\sigma}(W_1)\bar{\sigma}(W_2) \end{cases}$$



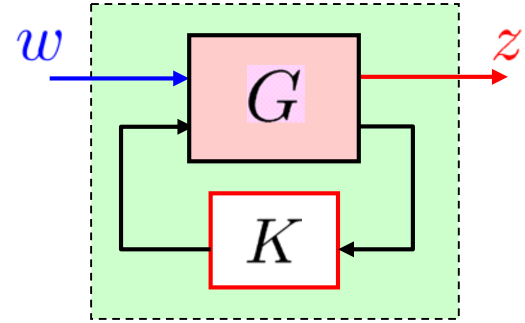
Generalized Weighted Formulation



$$F_l(G,K) = \begin{bmatrix} \bar{W}_1 K \\ \bar{W}_2 \end{bmatrix} (I - GK)^{-1} \begin{bmatrix} \bar{W}_3 & G\bar{W}_4 \end{bmatrix}$$

$$= \begin{bmatrix} K_s \\ I \end{bmatrix} (I - G_S K_S)^{-1} \begin{bmatrix} I & G_S \end{bmatrix}$$

$$z = F_l(G,K)w$$



$$G_S = \bar{W}_2 G \bar{W}_4, \;\; K = \bar{W}_1^{-1} K_S \bar{W}_3^{-1}$$

$$\bar{W}_1 = W_1^{-1}, \;\; \bar{W}_2 = W_2, \;\; \bar{W}_3 = W_2^{-1}, \;\; \bar{W}_4 = W_1$$

$\bar{W}_1 = 0, \bar{W}_4 = 0$: minimum sensitivity problem

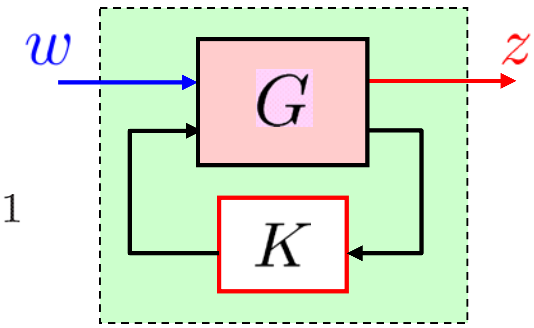
$\bar{W}_2 = 0, \bar{W}_4 = 0$: robust control problem for additive uncertainty

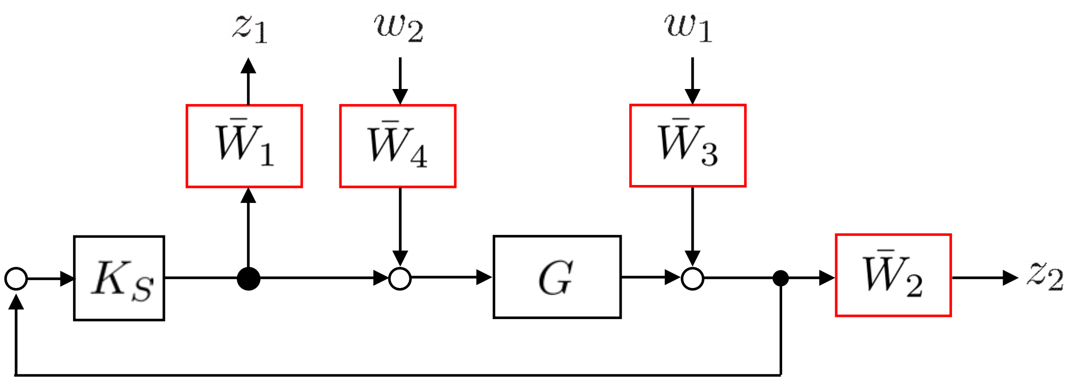
$\bar{W}_4 = 0$: mixed sensitivity problem

Robust Performance

$$\|F_l(G,K)\|_\infty$$
$$= \left\| \begin{bmatrix} K_s \\ I \end{bmatrix} (I - G_S K_S)^{-1} \begin{bmatrix} I & G_S \end{bmatrix} \right\|_\infty$$
$$= \left\| \begin{bmatrix} K_s S_s & K_s S_s G_s \\ S_s & S_s G_s \end{bmatrix} \right\|_\infty \quad S_s = (I - G_S K_S)^{-1}$$

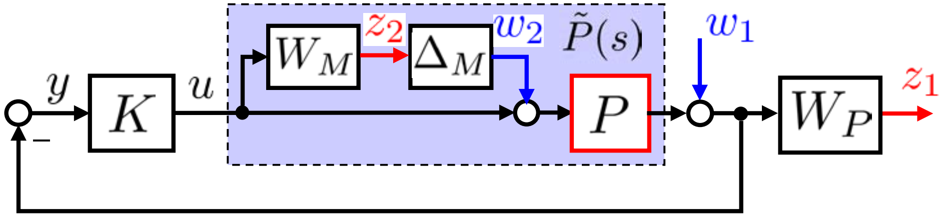
$$z = F_l(G,K)w$$





RP: Robust Performance

$$F_l(G,K)$$
$$= \begin{bmatrix} -W_M T_I & -W_M K S_o \\ W_P S_o P & W_P S_o \end{bmatrix}$$



Advantage of LSDP [SP05, p. 372]

(1) LSDP is relatively easy to use, based on classical loop-shaping ideas

(2) There exists a closed formula for the H_∞ optimal cost $\gamma_{\min}$, which in turn corresponds to a maximum stability margin $\epsilon_{\max} = \gamma_{\min}^{-1}$

(3) No γ -iteration is required in the solution γ_{opt} .

[hinfsyn/dksyn : γ -iteration]

(4) In case a process has a pole on the imaginary axis, LSDP does not require the additional operation to solve the problem.

hinfsyn : Assumptions

(A1) (A, B_2) is stabilizable and (C_2, A) is detectable

(A2) (A, B_1) is controllable and (C_1, A) is observable

Full rank on the imaginary axis

(5) Except for special systems, ones with all-pass factors, there are no pole-zero cancellations between the plant and controller

(6) LSDP can permit a wider error of the model

MATLAB Command loopsyn

H_∞ optimal controller synthesis for LTI plant

[K, Cl, Gam, info] = loopsyn(G, Gd, RANGE)

Input argument

G	(Generalized) LTI Plant
Gd	Desired Loop-shape (LTI model)
(option) RANGE	Desired frequency range for loop-shaping
	$\{\omega_{\min}, \omega_{\max}\}, \ 10\omega_{\min} < \omega_{\max}$ (Default) $\{0, \infty\}$

Output argument

K	LTI Controller
Cl	LTI Closed-loop system
Gam	Loop-shaping accuracy $\gamma \geq 1$. $\gamma = 1$: perfect fit.
Info	Information of output result

(option) Info.W, W satisfying $\sigma(G_d(j\omega)) \approx \sigma(G(j\omega)W(j\omega)), \ \forall\omega$

Info.Gs, $G_s = GW$

Info.Ks, $K_s = WK$

Info.range $\{\omega_{\min}, \omega_{\max}\}$

MATLAB Command ncfsyn

Loop shaping design using Glover-McFarlane method

[K, Cl, Gam, info] = ncfsyn(G, W1, W2, ‘ref’)

Input argument

G	(Generalized) LTI Plant
W1, W2	Weights
(option) ‘ref’	Compute normalized coprime factor loop-shaping controller

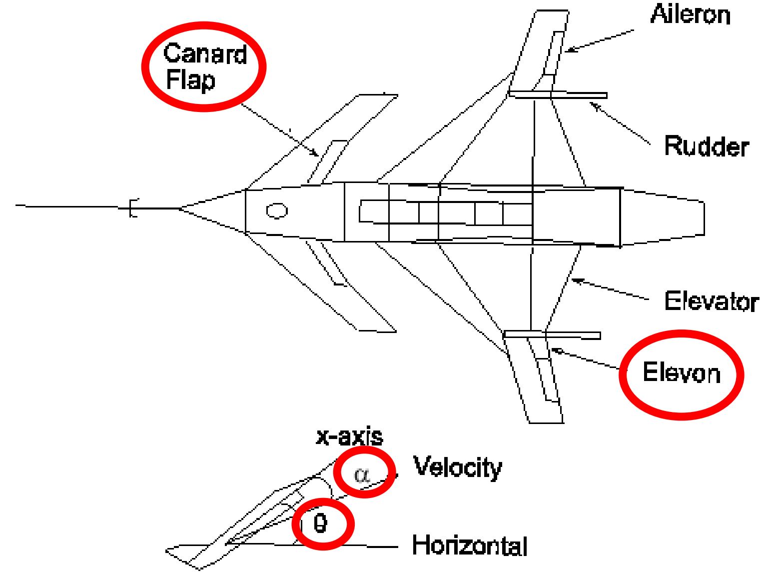
Output argument

K	LTI Controller
Cl	LTI Closed-loop system
Gam	Loop-shaping accuracy $\gamma \geq 1$. $\gamma = 1$: perfect fit.
Info	Information of output result

15~17

The 2-by-2 NASA HIMAT aircraft model

(Loop Shaping of HIMAT Pitch Axis Controller)



$$x^T = \begin{pmatrix} \dot{\alpha} & \alpha & \dot{\theta} & \theta & x_e & x_c \end{pmatrix}$$

$$u^T = \begin{pmatrix} \delta_e & \delta_c \end{pmatrix} \quad y^T = \begin{pmatrix} \alpha & \theta \end{pmatrix}$$

α : angle of attack

θ : attitude angle

δ_e : elevon actuator

δ_c : canard actuator

HIMAT: Nominal Plant Model

State Space Form (Matrix Representation)

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \qquad x^T = \begin{pmatrix} \dot{\alpha} & \alpha & \dot{\theta} & \theta & x_e & x_c \end{pmatrix}$$

$$u^T = \begin{pmatrix} \delta_e & \delta_c \end{pmatrix} \quad y^T = \begin{pmatrix} \alpha & \theta \end{pmatrix}$$

$$A = \begin{pmatrix} -0.023 & -36.62 & -18.90 & -32.09 & 3.251 & -0.763 \\ 0.000 & -1.900 & -0.983 & -0.000 & -0.171 & -0.005 \\ 0.012 & 11.72 & -2.632 & 0.000 & -31.60 & 22.40 \\ 0 & 0 & 1.000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -30.00 & 0 \\ 0 & 0 & 0 & 0 & 0 & -30.00 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 30 & 0 \\ 0 & 30 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

MATLAB Command

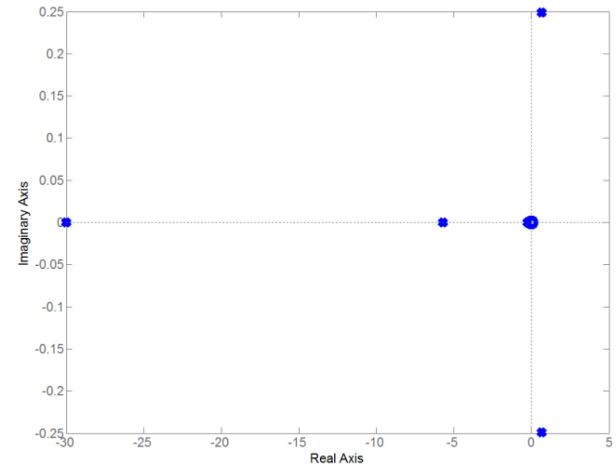
```
% NASA HiMAT model G(s)
ag =[ -2.2567e-02 -3.6617e+01 -1.8897e+01 -3.2090e+01 3.2509e+00 -7.6257e-01;
        9.2572e-05 -1.8997e+00  9.8312e-01  -7.2562e-04 -1.7080e-01 -4.9652e-03;
        1.2338e-02   1.1720e+01 -2.6316e+00  8.7582e-04 -3.1604e+01 2.2396e+01;
                 0            0   1.0000e+00           0           0          0;
                 0            0            0           0 -3.0000e+01          0;
                 0            0            0           0           0 -3.0000e+01];
bg = [ 0 0; 0 0; 0 0; 0 0; 30 0; 0 30];
cg = [ 0 1 0 0 0 0; 0 0 0 1 0 0];
dg = [ 0 0; 0 0];
G=ss(ag,bg,cg,dg);
```

HIMAT: Nominal Plant Model

Controllability ○

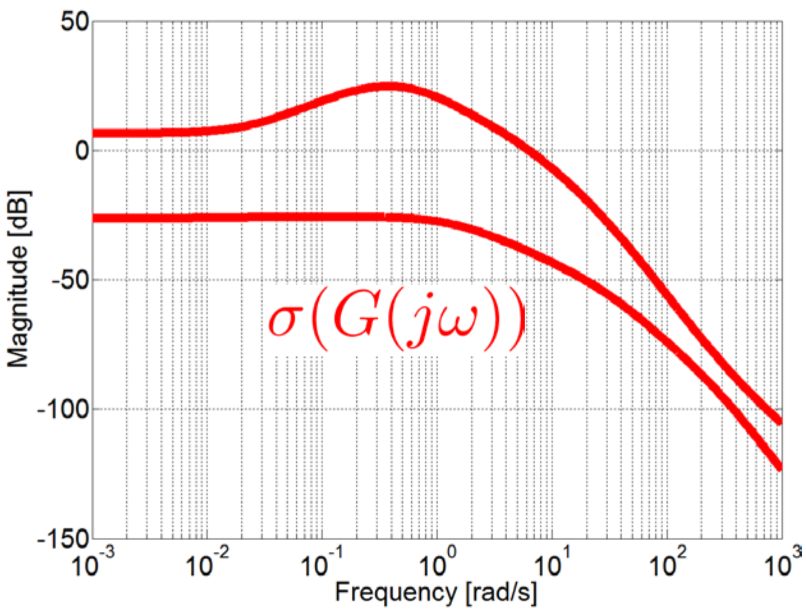
Observability ○

Poles (Stability) -5.6757, -0.2578, -30, -30,
0.6898 ± 0.2488i
Unstable

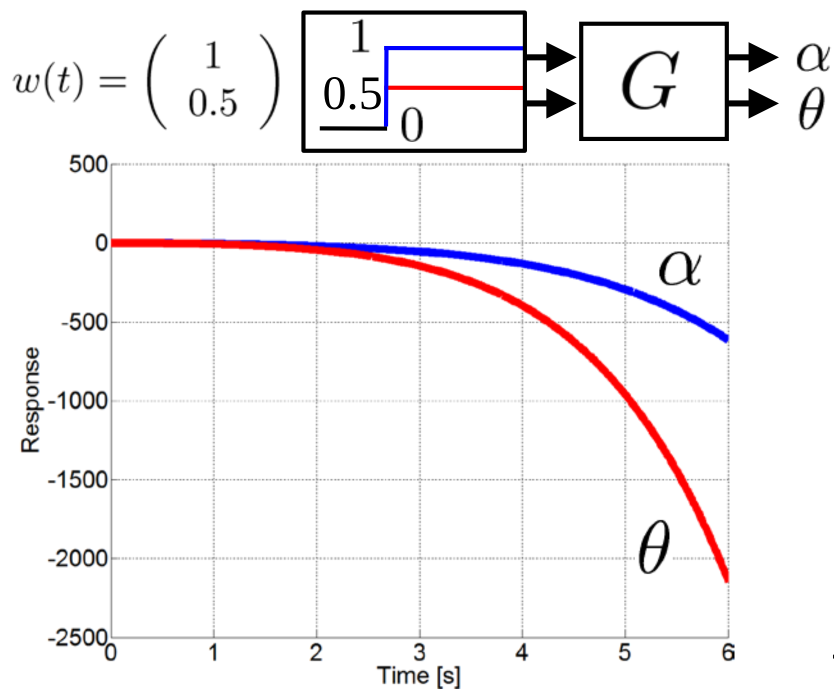


Zeros -0.0210

Frequency Response

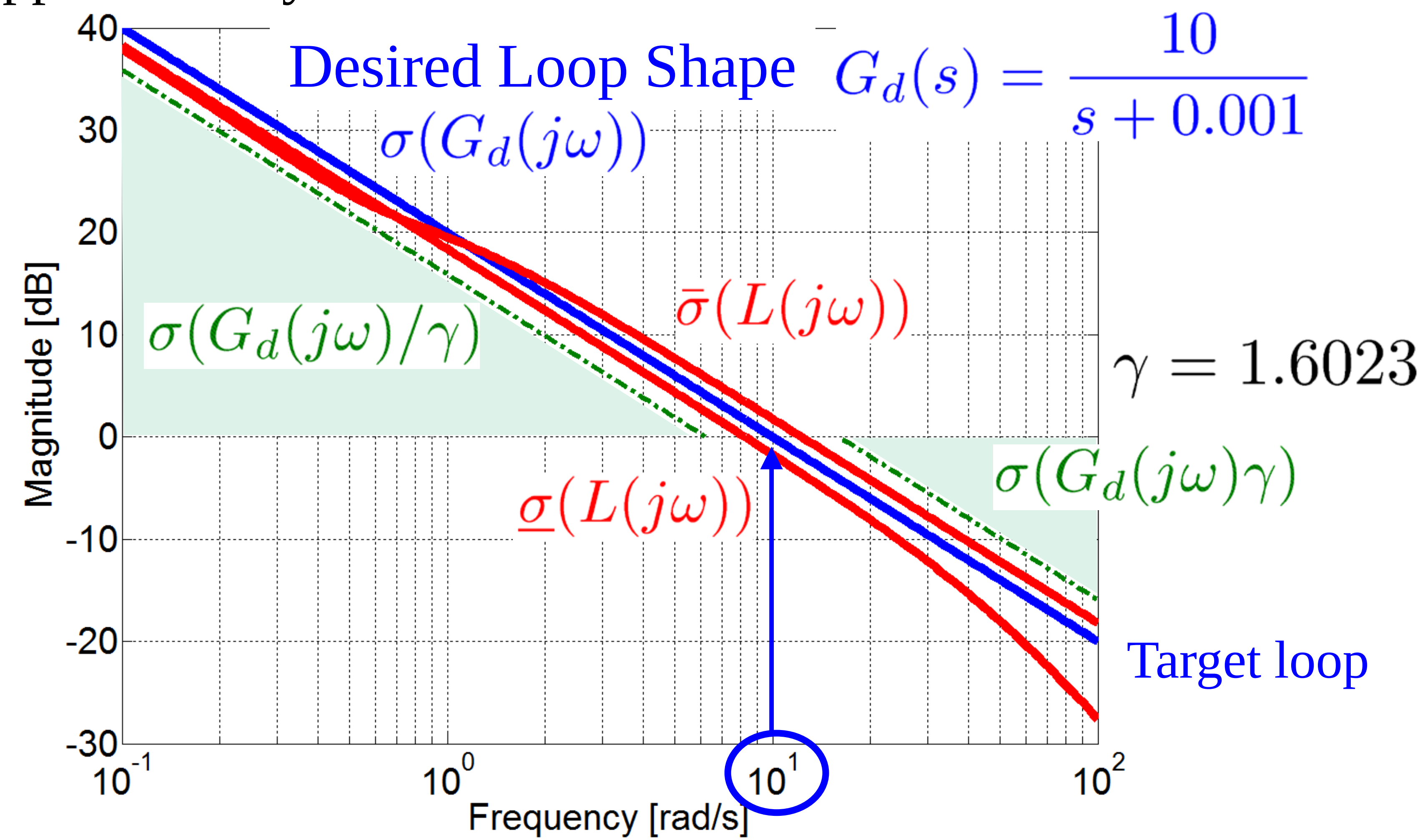


Step Response for Nominal Plant Model



HIMAT: Specifications and Open-loop

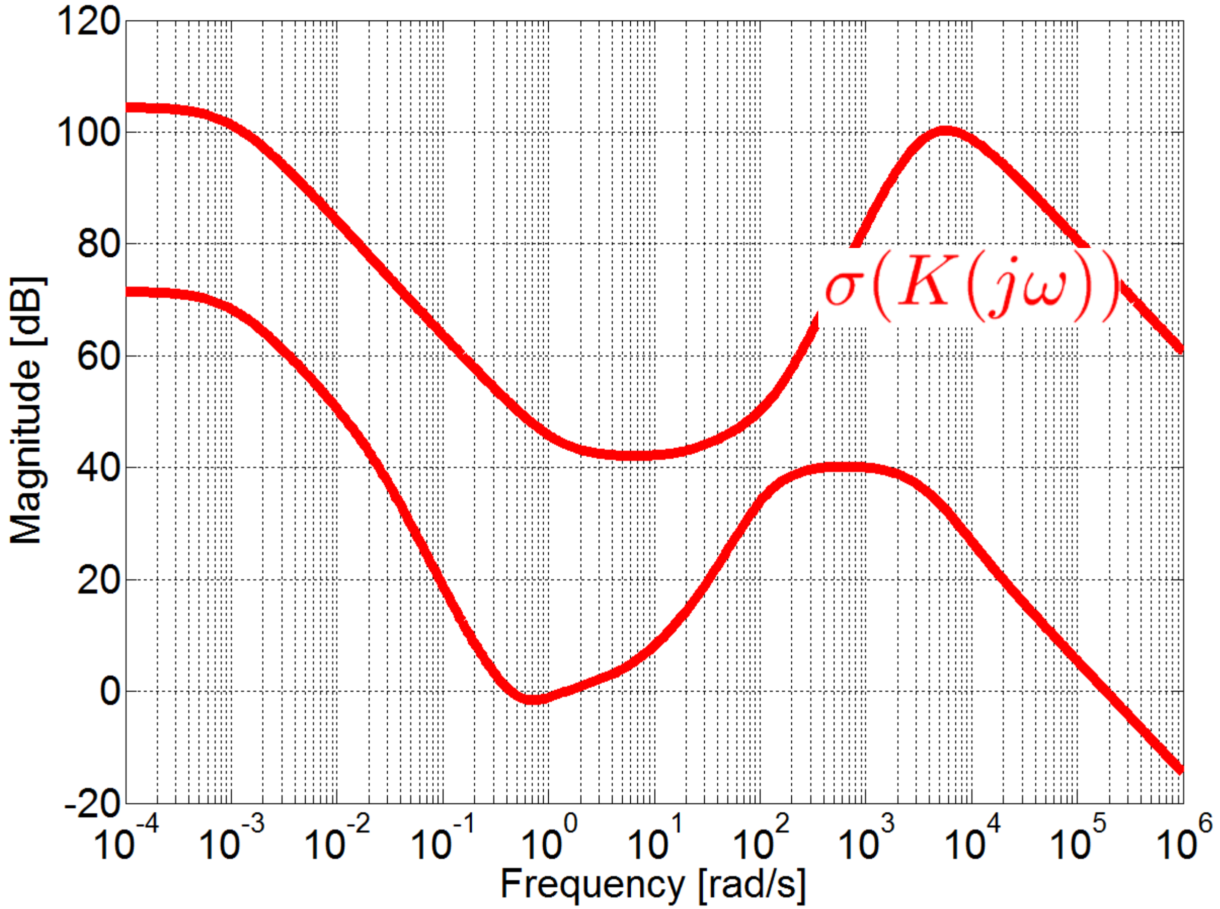
Approximately a bandwidth of 10 rad/s



MATLAB Command

```
Gd= tf( 10, [1 .001] ) ;
[K,CL,GAM,INFO]=loopsyn(G,Gd);
sigma(Gd,'b',G*K,'r',Gd/GAM,'g:',Gd*GAM,'g:',{.1,100})
```

HIMAT: Controller



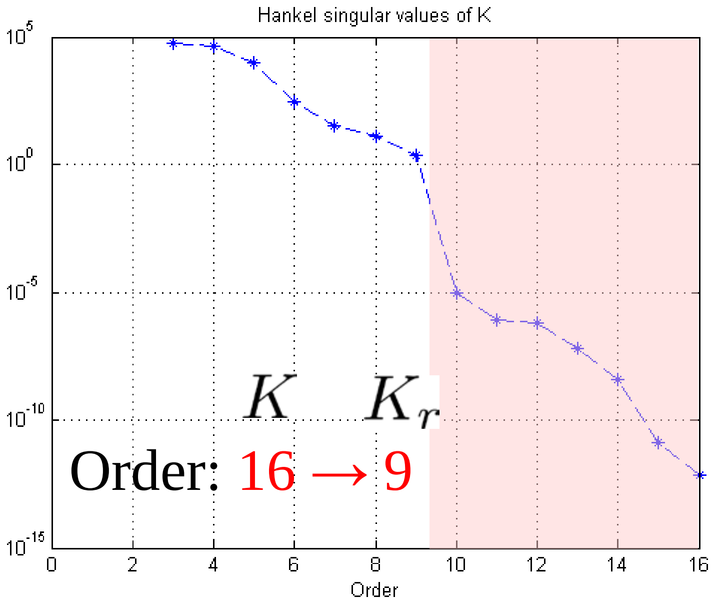
MATLAB Command
sigma(K);

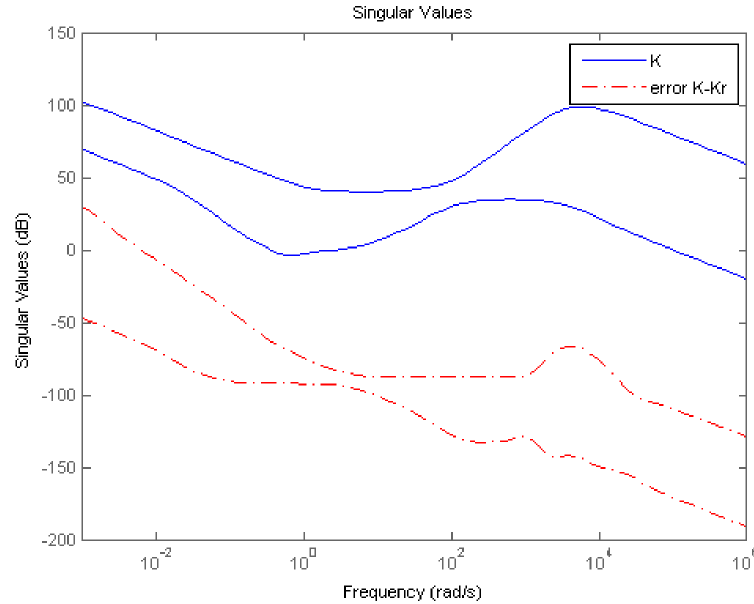
Order: 16

Numerical problems or inaccuracies may be caused too high order

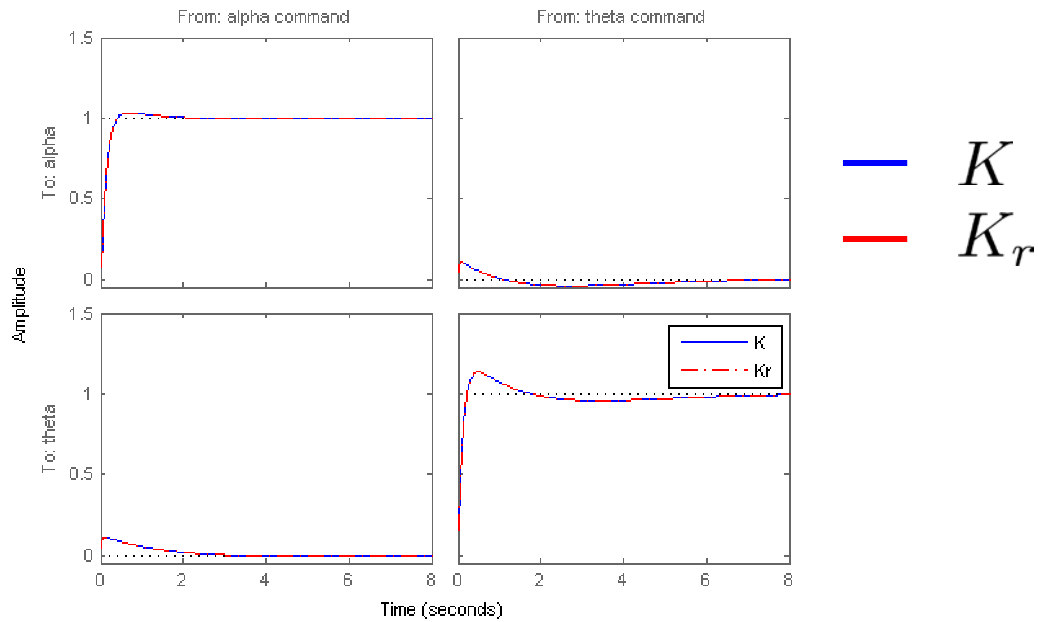
➡ Difficult to implement

HIMAT: Controller Model Reduction





Step responses



MATLAB Command

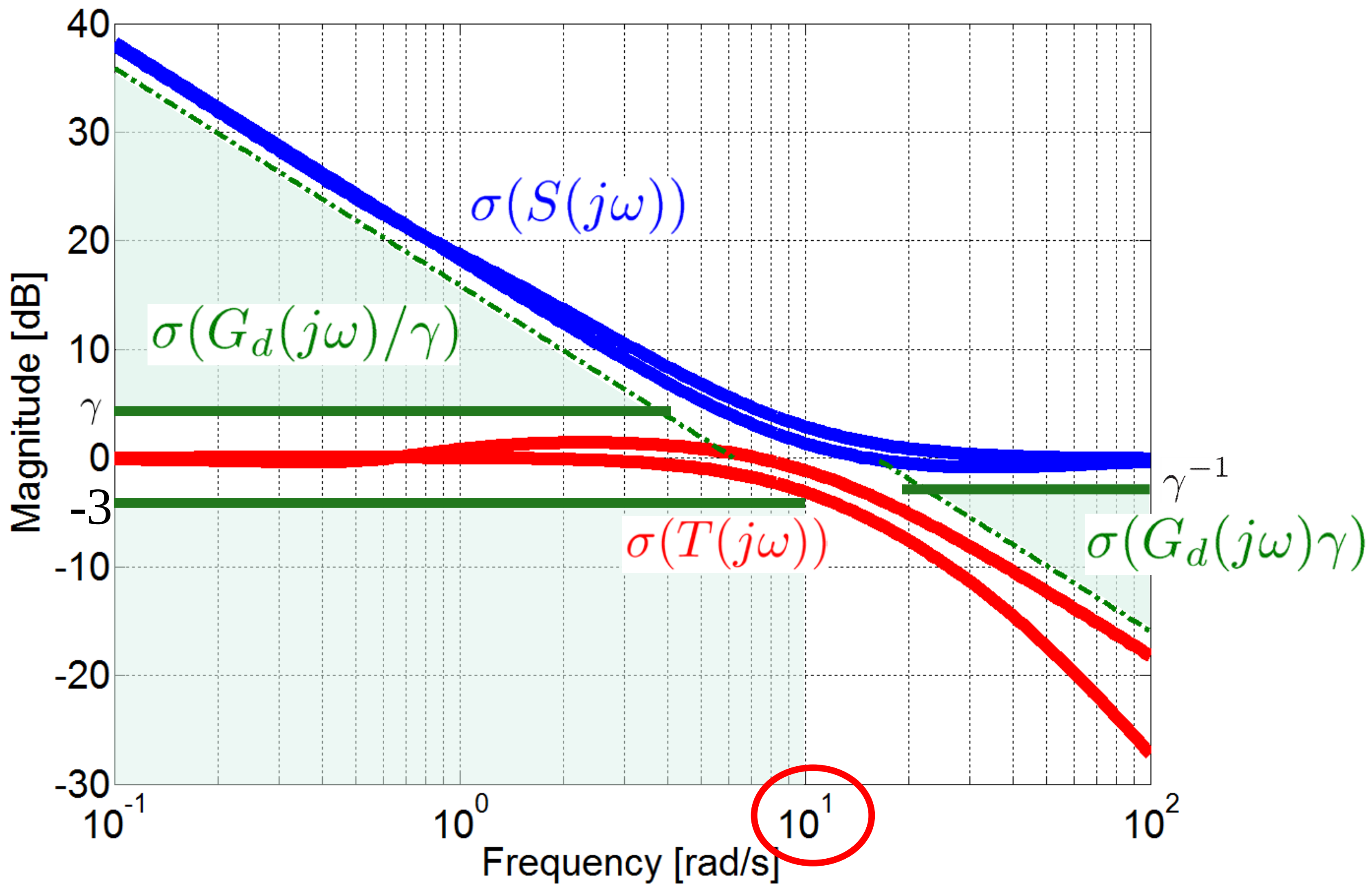
```
size(K)

hsv = hankelsv(K);
semilogy(hsv,'*--'), grid
title('Hankel singular values of K'),
xlabel('Order')

Kr = reduce(K,9); order(Kr)

sigma(K,'b',K-Kr,'r-.')
legend('K','error K-Kr')
```

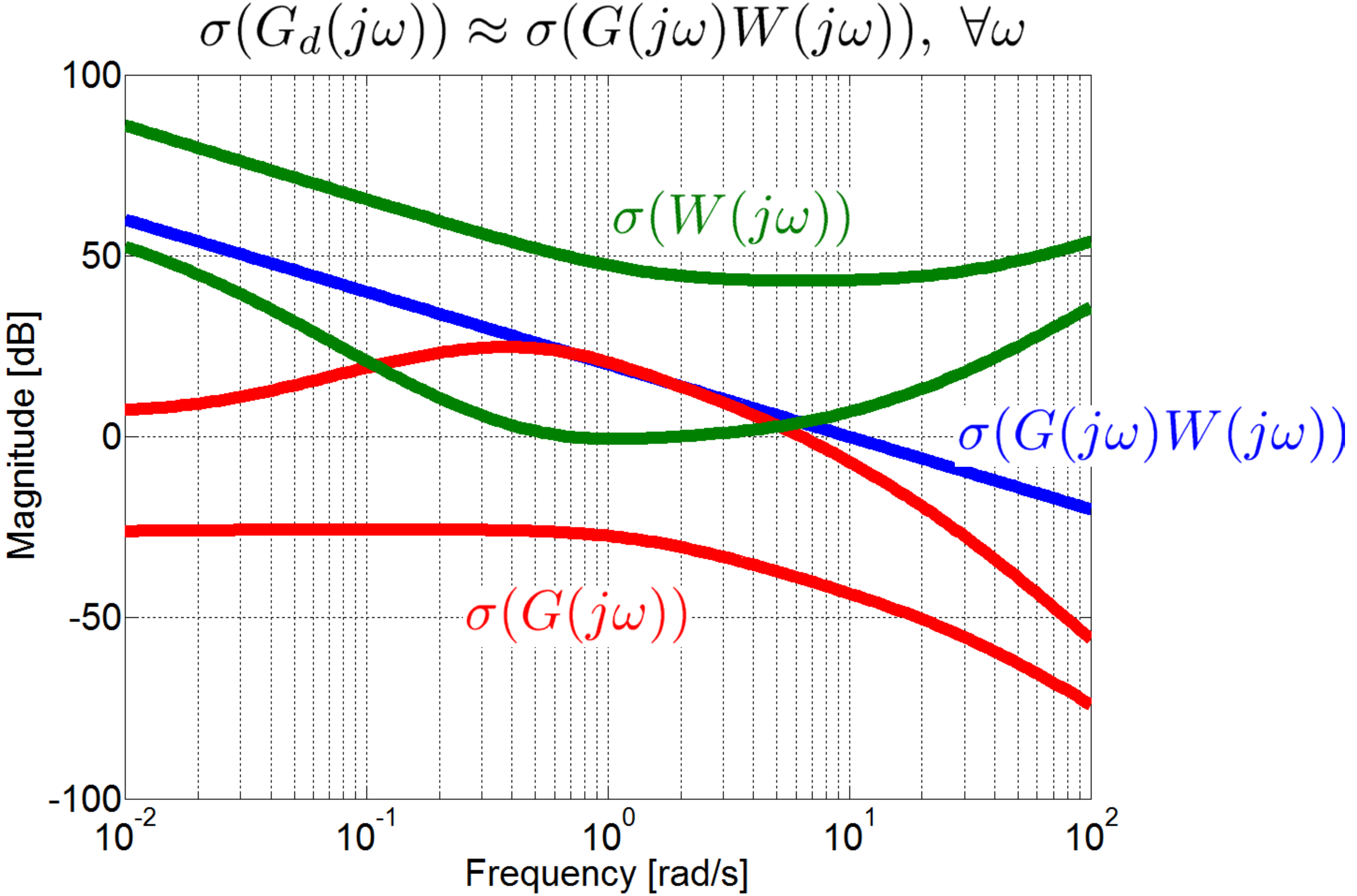
HIMAT: Sensitivity



MATLAB Command

```
T = feedback(G*K,eye(2));
S = eye(2)-T;
sigma(inv(S),'m',T,'g',L,'r--',Gd,'b',Gd/GAM,'b:',Gd*GAM,'b:',{.1,100})
```

HIMAT: Designed Weight

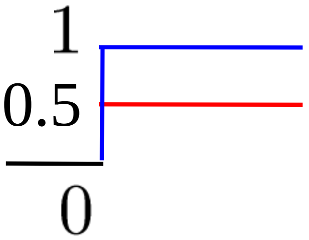


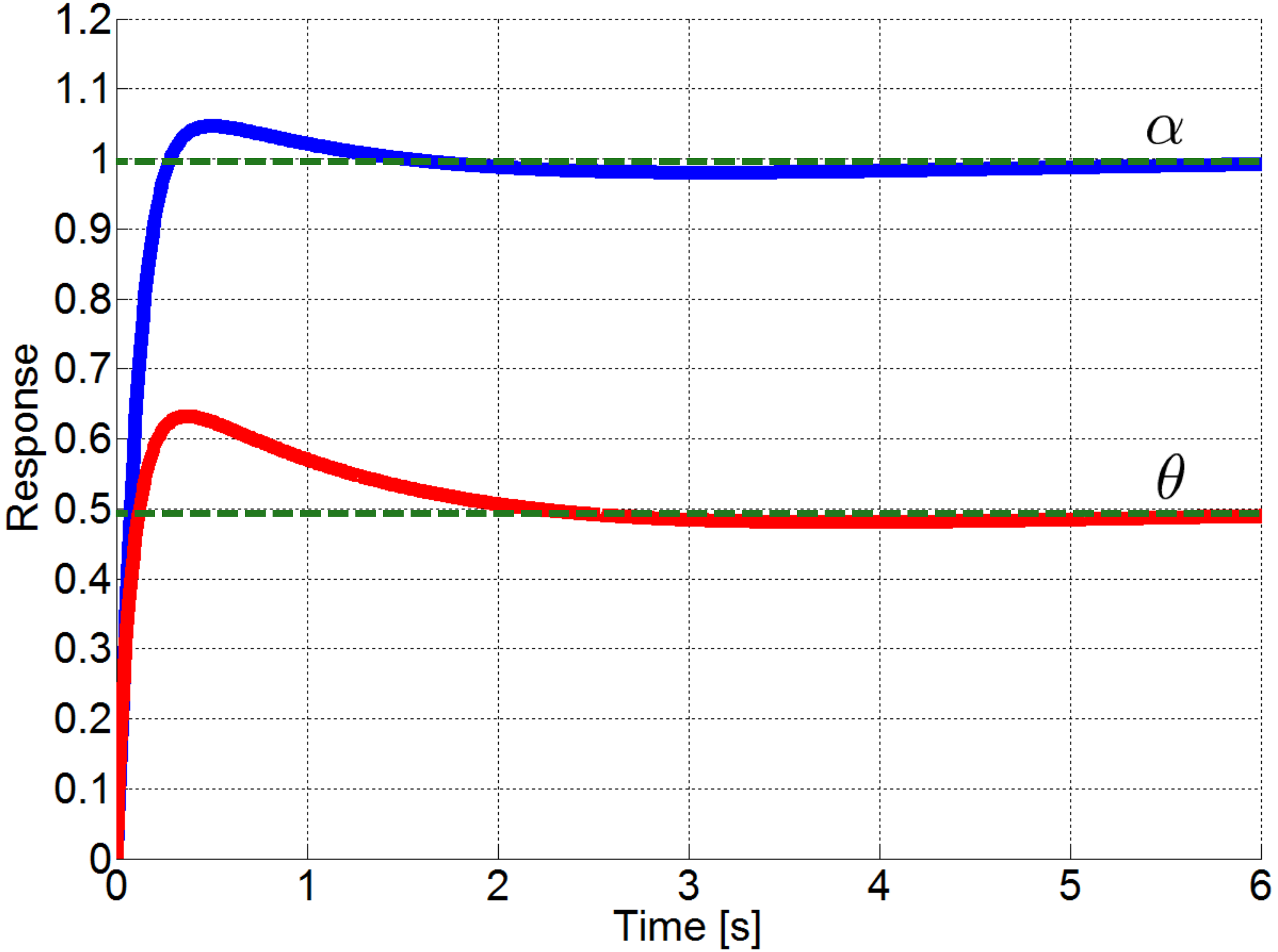
MATLAB Command

```
sigma(INFO.Gs,‘b',G,'r',INFO.W,‘g',{.01,100});
```

HIMAT: Step Response (closed loop)

$$w(t) = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}$$





Youla Parameterization

Case 2: Unstable Plant $P(s)$ [SP05, p. 149]

Coprime Factorization [SP05, p. 122]

$$P(s) = \frac{N(s)}{M(s)}$$

Coprime: No common right-half plane(RHP) zeros

$N(s), M(s)$: Proper Stable Transfer Functions

[SP05, Ex. 4.1] $P(s) = \dfrac{(s-1)(s+2)}{(s-3)(s+4)}$

$$\Rightarrow N(s) = \frac{s-1}{s+4}, \quad M(s) = \frac{s-3}{s+2} \qquad (*)$$

Bezout Identity $NX + MY = 1 \longleftrightarrow M(s), N(s)$: Coprime

$X(s), Y(s)$: Proper Stable Transfer Functions

[SP05, Ex.] $M(s), N(s)$: (*) $\Rightarrow X(s) = \dfrac{s+32}{2s+4}, \quad Y(s) = \dfrac{s-16}{2s+8}$

[Ex.] $5x + 3y = 1$ x, y: Integer $\Rightarrow x = -1 - 3q, \quad y = 2 + 5q$

q : Integer

Youla Parameterization

Case 2: Unstable Plants $P(s)$ [SP05, p. 149]

A Stabilizing Controller $K(s) = \frac{X(s)}{Y(s)}$ $(Q(s) = 0)$

[SP05, Ex.] $P(s) = \frac{(s-1)(s+2)}{(s-3)(s+4)}, \quad X(s) = \frac{s+32}{2s+4}, \quad Y(s) = \frac{s-16}{2s+8}$

$$\Rightarrow \quad K(s) = \frac{X(s)}{Y(s)} = \frac{s^2+36s+128}{s^2-14s-32}$$

All Stabilizing Controllers $K(s) = \frac{X(s)+M(s)Q(s)}{Y(s)-N(s)Q(s)}$

$$\left(N = P, M = 1, X = 0, Y = 1 \Rightarrow K = \frac{Q}{1-PQ} \right)$$

Gang of Four

$$S = \underline{M(Y-NQ)} \qquad T = \underline{N(X+MQ)}$$

$$PS = \underline{N(Y-NQ)} \qquad KS = \underline{M(X+MQ)}$$

Affine Functions of Q

Coprime Factorization: State-space Procedure

Observer design

$$\begin{cases} \dot{x} = Ax + Bu - L(y - Cx) \\ y = Cx \end{cases}$$

$$y = C(sI - A - LC)^{-1} \begin{bmatrix} B & -L \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}$$

$$= \begin{bmatrix} N_l & I - M_l \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \quad \Rightarrow \quad y = M_l(s)^{-1} N_l(s) u$$

State feedback pole assignment

$$\begin{cases} \dot{x} = (A + BF)x + Be \\ e = u + z, \ \ z = -Fx \\ y = Cx \end{cases}$$

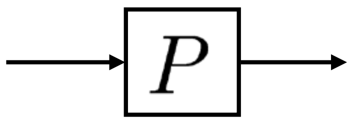
$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} C \\ -F \end{bmatrix} (sI - A - BF)^{-1} Be = \begin{bmatrix} N_r \\ I - M_r \end{bmatrix} e$$

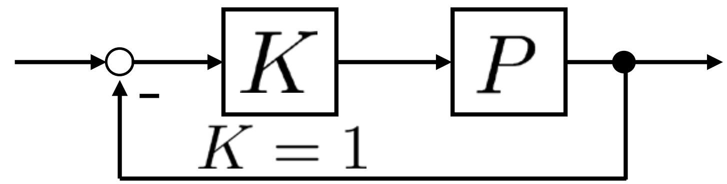
$$\Rightarrow \quad e = u + (I - M_r(s))e = M_r(s)^{-1} u$$

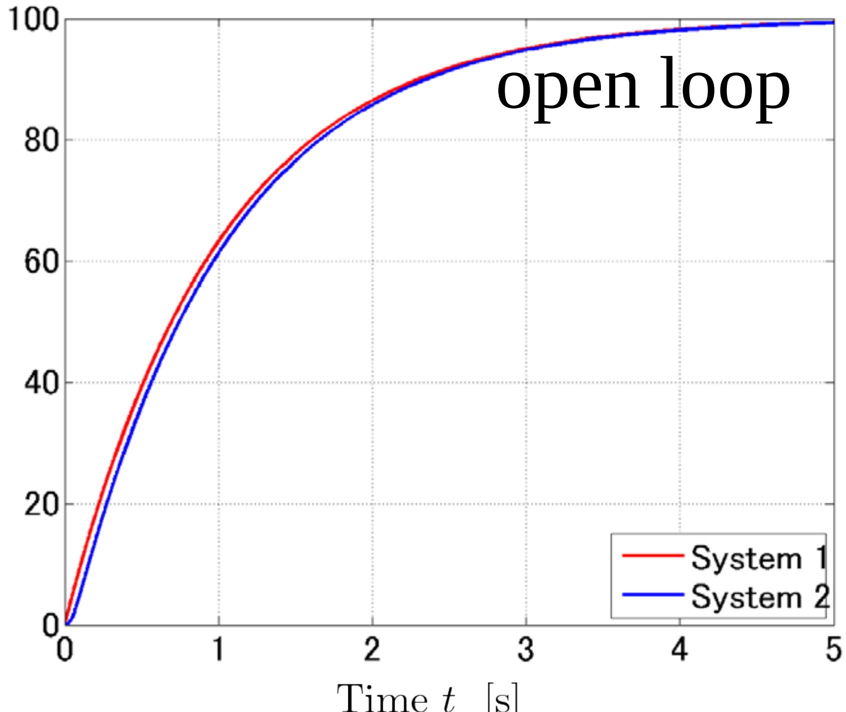
$$\Rightarrow \quad y = N_r(s)e = N_r(s) M_r(s)^{-1} u$$

When Are Two Systems Similar ? [AM09, pp. 349-352]

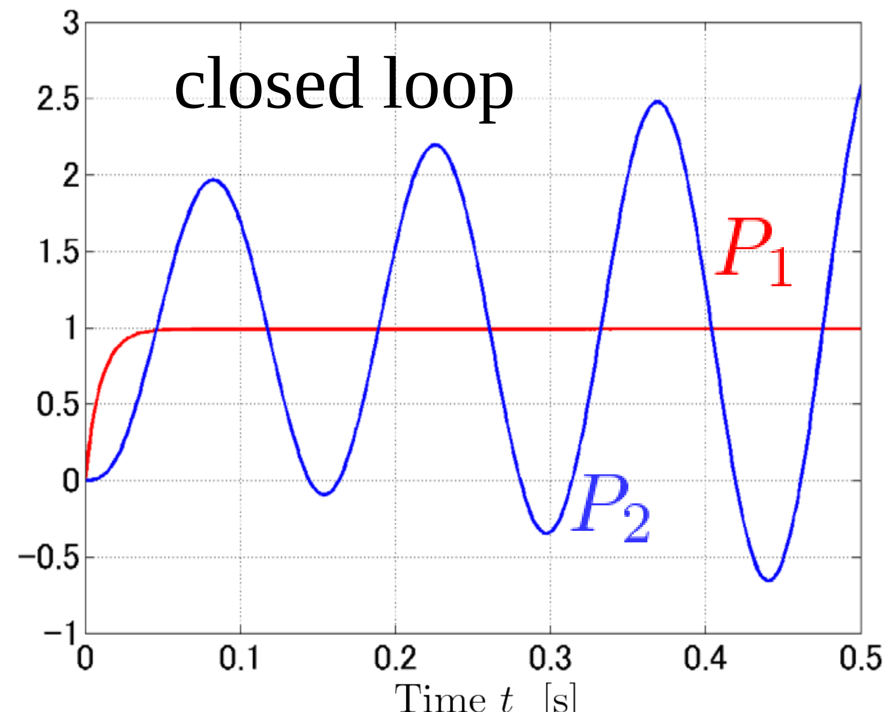
[AM09, Ex 12.2] $P_1(s) = \dfrac{100}{s+1}$, $P_2(s) = \dfrac{100}{(s+1)(0.025s+1)^2}$







(a) Step response (open loop)



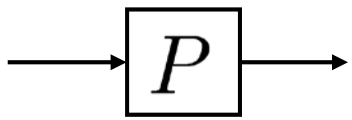
(b) Step response (closed loop)

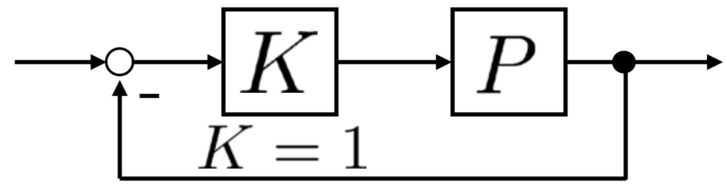
Similar in Open Loop but Large Differences in Closed Loop

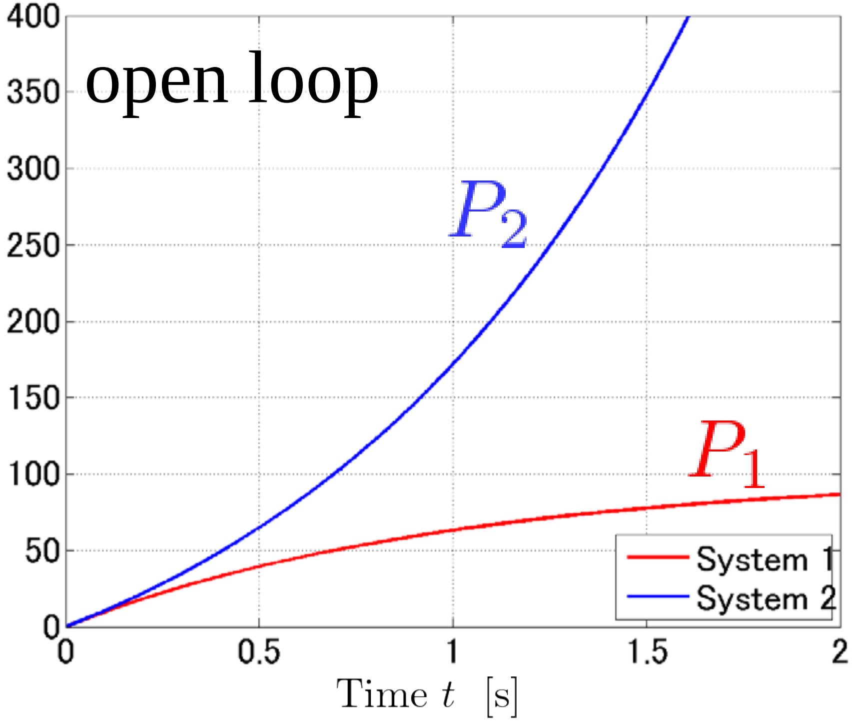
[AM09] K.J. Astrom and R.M. Murray, *Feedback Systems*, Princeton Univ. Press, 2009.

When Are Two Systems Similar ? [AM09, pp. 349-352]

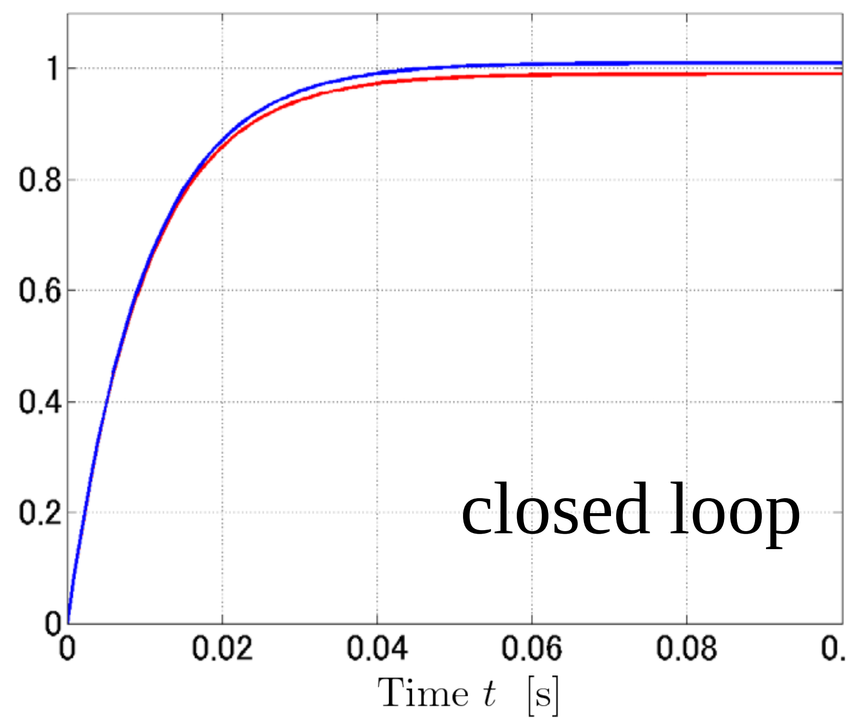
[AM09, Ex 12.3] $P_1(s) = \dfrac{k}{s+1}, \quad P_2(s) = \dfrac{k}{(s-1)} \qquad k = 100$







(a) Step response (open loop)



(b) Step response (closed loop)

Different in Open Loop but Similar in Closed Loop

Vinnicombe Metric (ν -gap Metric) [Zhou98, Chap.17]

$\delta_g(G_o, G_1)$: the smallest value of $\left\| \left[\Delta_N(j\omega) \;\; \Delta_M(j\omega) \right] \right\|_\infty$
that perturbs G_o into G_1 is called the *gap* between G_o and G_1

If $\delta_g(G_o, G_1) < b(G_o, K)$, then the closed loop system
with G_1 and K will also be stable

$\delta_\nu(G_o, G_1)$: the ν -gap between G_o and G_1

If $\delta_\nu(G_o, G_1) < b(G_o, K)$,
then we have closed loop stability of G_1 and K

$b(G_o, K)$ gives the radius (in terms of the distance in the ν -gap metric)
of the largest "ball" of plants stabilized by K

Note: Both δ_g and δ_ν are metrics (i.e. distance measures)

(1) $0 \le \delta_\nu(G_o, G_1) \le 1$ (2) $\delta_\nu(G_o, G_1) = 0 \;\Rightarrow\; G_o = G_1$

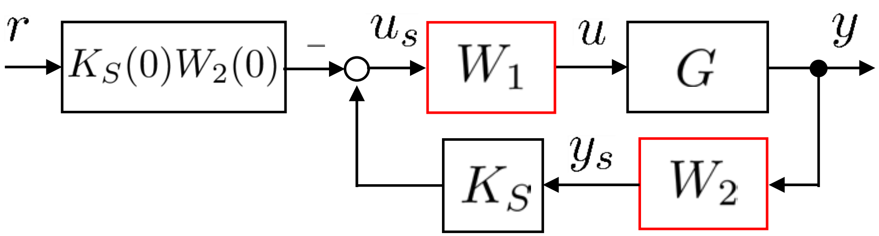
(3) $\delta_\nu(G_o, G_1) = \delta_\nu(G_1, G_o)$

(4) $\delta_\nu(G_0, G_2) \le \delta_\nu(G_0, G_1) + \delta_\nu(G_1, G_2)$ (Triangle inequality)

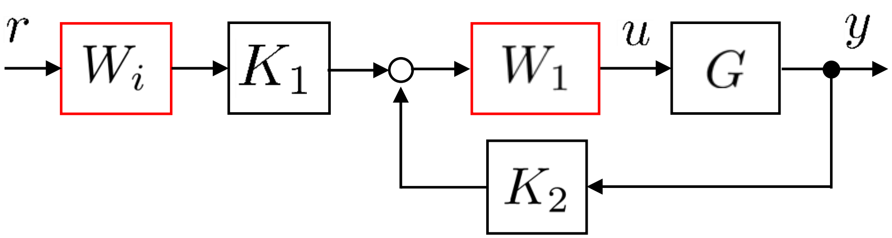
[Zhou98] K. Zhou with J.C. Doyle, *Essentials of Robust Control*, Prentice Hall, 1998.

Examples of NLCF Robust Control Problems

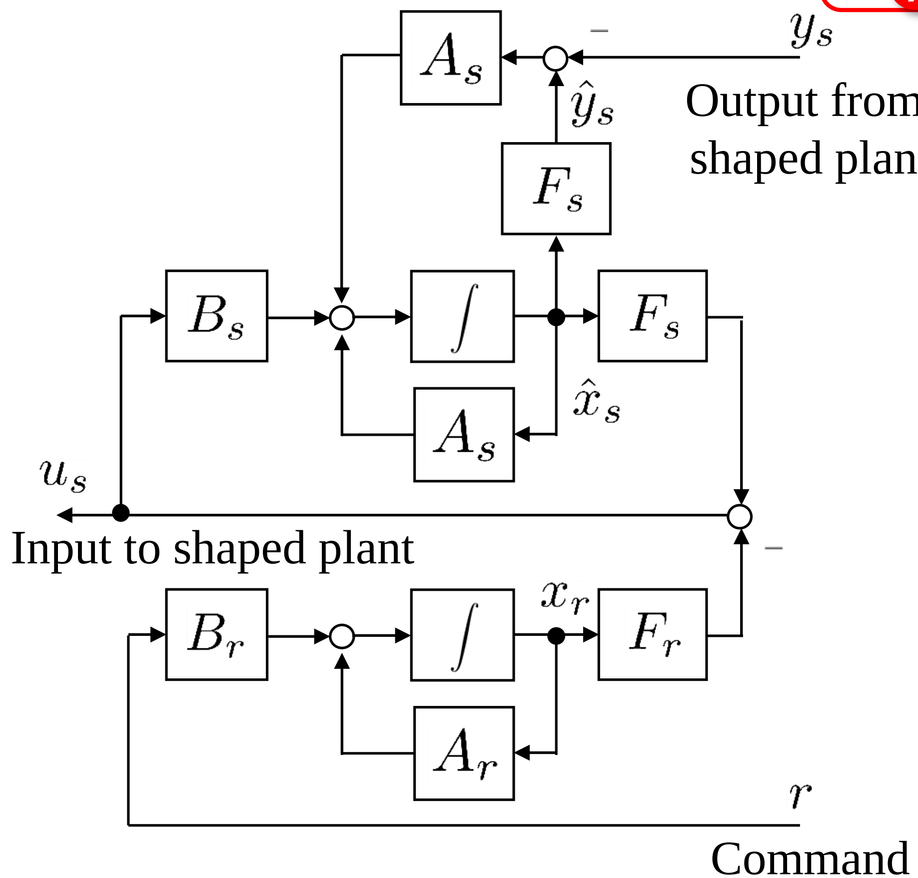
[SP05, pp. 370-381]



A Practical Implementation



Two-degrees of freedom
H_∞ loop-shaping controller



Observer-based Structure for
H_∞ loop-shaping controller

Implementation Issues [SP05, p. 380]

Discrete-time controllers, Anti-windup, Bumpless transfer

NLCF Robust Control Problem

Nominal Plant Model $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$

Sub-optimal Solution (H_∞ controller)

$X > 0$, $Z > 0$ satisfying that

Riccati equalities

$$X\tilde{A}_X + \tilde{A}_X^T X - XBS^{-1}B^T X + C^T R^{-1} C = 0$$
$$Z\tilde{A}_Z^T + \tilde{A}_Z Z - ZC^T R^{-1} CZ + BS^{-1}B^T = 0$$

where $\tilde{A}_X = A - BS^{-1}D^T C$, $\tilde{A}_Z = A - BD^T R^{-1} C$

$$R = DD^T, \;\; S = I + D^T D$$

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A + BF & -\gamma^2 W_1^{-T} BS^{-1/2} & \gamma^2 \zeta^{-1} W_1^{-T} ZC^T R^{-1/2} \\ \hline F & S^{-1/2} & \zeta^{-1} D^T R^{-1/2} \\ C + DF & DS^{-1/2} & -\zeta^{-1} R^{-1/2} \end{array}\right]$$

where $\zeta = \sqrt{\gamma^2 - 1}$, $W_1 = (1 - \gamma^2) I + XZ$,

$$F = -S^{-1}(D^T C + B^T X) \; , \; S = I + D^T D$$

Observer-based Structure

Central controller ($\Phi = 0$)

$$K = \left[\begin{array}{c|c} A + BF + \gamma^2 W_1^{-T} Z C^T (C + DF) & \gamma^2 W_1^{-T} X C^T \\ \hline B^T X & -D^T \end{array}\right]$$

$$\begin{cases} \dot{x}_k = A x_k + Bu + \hat{H}(y - C x_k - Du) \\ u = -B^T X x_k - D^T y \end{cases}$$

where $\hat{H} = (BD^T + V^{-1} C^T) R^{-1}$,

$V = Z^{-1} - \gamma^{-2}(X + Z^{-1})$

Features

Observer gain $\hat{H}$ is automatically designed

Observer gain $\hat{H}$ is related to both Z and X (Riccati Solutions)

$$\gamma \to \infty \quad \Rightarrow \quad \hat{H}_\infty = (BD^T + ZC^T) R^{-1}$$

Cost Function $J = \int_0^\infty (y^T \hat{Q} y + u^T \hat{R} u) dt$

$$\Rightarrow \quad W_1(s) = \hat{R}^{-1/2}(s), \; W_2(s) = \hat{Q}^{1/2}(s)$$

Computation

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \qquad G \text{ is stable} \quad \Leftrightarrow \quad \mathrm{Re}[\lambda(A)] < 0$$

$\mathcal{H}_2$ -norm Controllability/Observability

$\mathcal{H}_\infty$-norm Hamiltonian Matrix

$$H = \left[\begin{array}{c|c} A + BS^{-1}D^TC & BS^{-1}B^T \\ \hline -C^T(I + DS^{-1}D^T)C & -(A + BS^{-1}D^{-1}C)^T \end{array}\right]$$

$$s = \gamma^2 I - D^T D$$

Theorem

$$\|G(s)\|_\infty < \gamma \quad \Leftrightarrow \quad \begin{cases} \bar{\sigma}(D) < \gamma \\ H \text{ has no eigenvalues on imaginary axis} \end{cases}$$

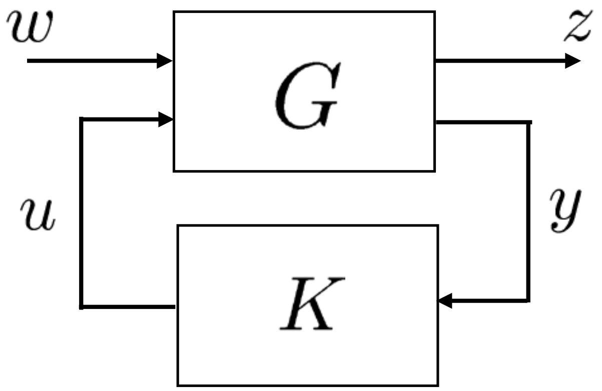
Graphical Test

$$\max_\omega \bar{\sigma}[G(j\omega)] < \gamma$$

H_∞ Frobenius synthesis with Hadamard weight

Given $\gamma > \gamma_{min}$, find all stabilizing controllers K such that

$$\|W \circ F_l(G, K)\|_{\infty F} \leq \gamma$$



F. van Diggelen and K. Glover, “A Hadamard weighted loop shaping design procedure,” Proc. 31st IEEE CDC, 2193-2198, 1992

Stability Margin

$$b(G,K) := \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} \begin{bmatrix} I & G \end{bmatrix} \right\|_\infty^{-1}$$

The closed loop will be stable for all $\left\| \begin{bmatrix} \Delta_N & \Delta_M \end{bmatrix} \right\|_\infty < \epsilon \Leftrightarrow b(G,K) \geq \epsilon$

$b(G,K) > 0.2 \sim 0.3$ (for good robustness)

In SISO Systems

$$GM \geq \frac{1 + b(G,K)}{1 - b(G,K)}, \quad PM \geq 2\arcsin(b(G,K))$$

Proof:

$$\bar{\sigma}^2 \left\{ \begin{bmatrix} K \\ 1 \end{bmatrix} (1 - GK)^{-1} \begin{bmatrix} 1 & G \end{bmatrix} \right\}$$

$$= (1 + |K|^2)|1 - GK|^{-2}(1 + |G|^2) \leq \frac{1}{b^2(G,K)}, \ \forall \omega$$

$$\therefore \ b^2(1 + |K|^2)(1 + \frac{GM^2}{|K|^2}) \leq |1 - GM|^2$$

$\Rightarrow \ b^2(1 + GM)^2 \leq (1 - GM)^2$ for $0 \leq GM \leq 1$

LSDP in SISO Systems

For Low frequencies

$\underline{\sigma}(WG) \gg 1$

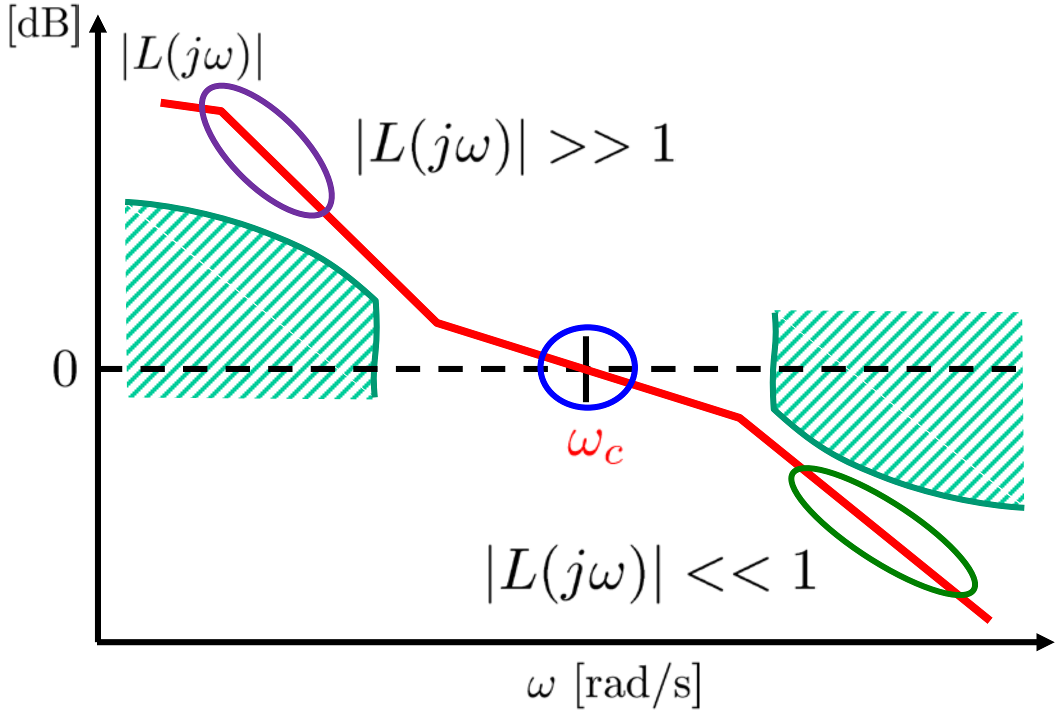
$$\Rightarrow \begin{cases} \left|\dfrac{1}{1-GK}\right| \leq \dfrac{\gamma}{|G||W|}, \\ \left|\dfrac{G}{1-GK}\right| \leq \dfrac{\gamma}{|W|} \end{cases}$$

For High frequencies

$\bar{\sigma}(WG) \ll 1$

$$\Rightarrow \begin{cases} \left|\dfrac{K}{1-GK}\right| \leq \gamma|W|, \\ \left|\dfrac{GK}{1-GK}\right| \leq \gamma|G||W| \end{cases}$$

$W := W_2 W_1$



Robust Performance in the ν -Gap Metric

$$\left.\begin{array}{l} G_0 = M^{-1}N \\ G_1 = (M+\Delta_M)^{-1}(N+\Delta_N) \\ \qquad\qquad \left\|\begin{bmatrix} \Delta_N & \Delta_M \end{bmatrix}\right\|_\infty < \beta \end{array}\right\} \delta_\nu(G_0, G_1) < \beta$$

If K stabilizes G_o with $b(G_0, K) \geq \beta$ then K will also stabilize G_1

A bound on the robust performance

$$\arcsin(b(G_1, K_1)) \geq \arcsin(b(G_0, K_0)) - \arcsin(\delta_\nu(G_1, G_0)) - \arcsin(\delta_\nu(K_1, K_0))$$

(The derivation of this is due to Vinnicombe and is non-trivial)

This inequality is a slightly stronger inequality than

$$b(G_1, K_1) \geq b(G_0, K_0) - \delta_\nu(G_1, G_0) - \delta_\nu(K_1, K_0)$$

Perturbed performance	Nominal performance	Plant perturbation	Controller perturbation

which is also true and shows clearly how the performance can be degraded by perturbations to the plant and controller

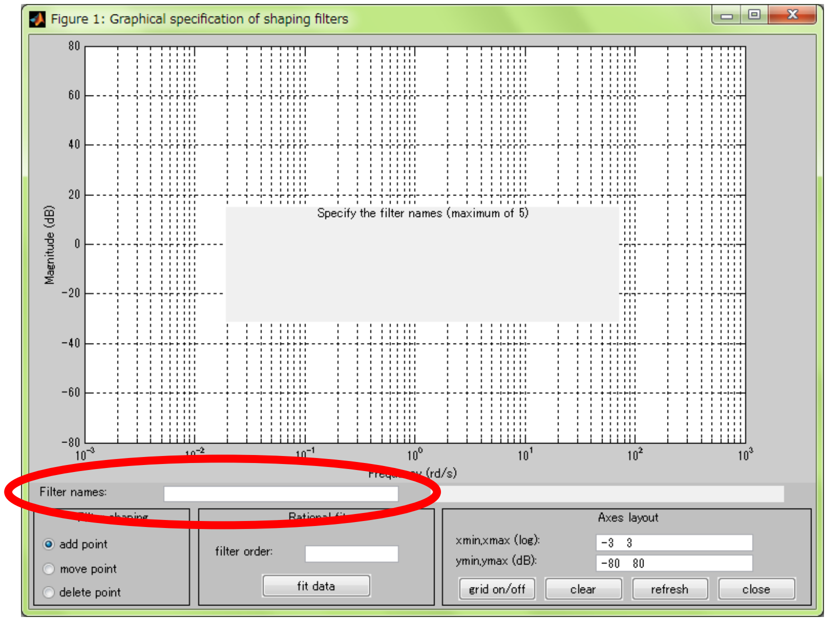
MATLAB command: “magshape” (graphical user interface)

Design Method

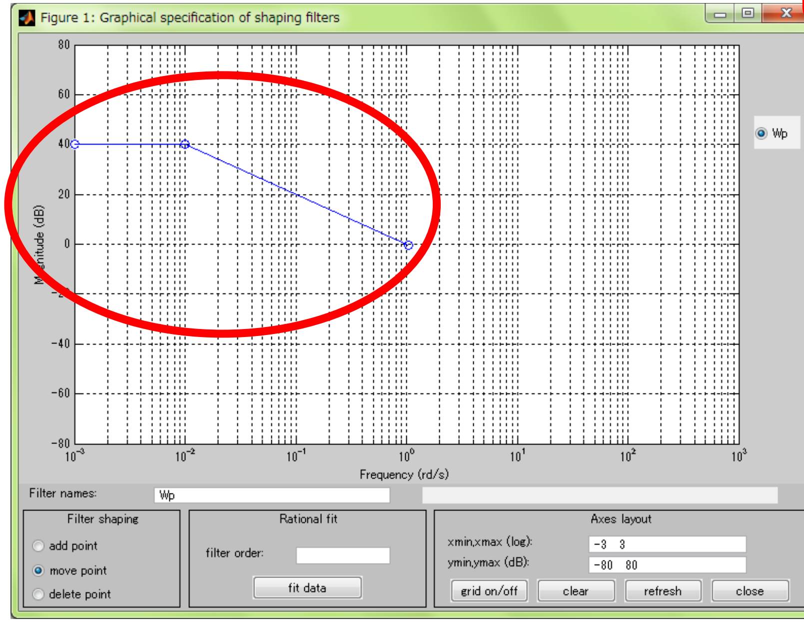
1. Express the design specifications in terms of loop shapes and shaping filters.
2. Specify the shaping filters by their magnitude profile. This is done interactively with the graphical user interface magshape.
3. Specify the control loop structure with the functions sconnect and smult, or alternatively with Simulink.
4. Solve the resulting H_∞ problem with one of the H_∞ synthesis functions.

To impose a given roll-off rate in the open-loop response, it is often desirable to use nonproper shaping filters. Meanwhile, “magshape” approximates them by high-pass filters. A drawback of this approximation is the introduction of fast parasitic modes in the filter and augmented plant realizations, which in turn may cause numerical difficulties.

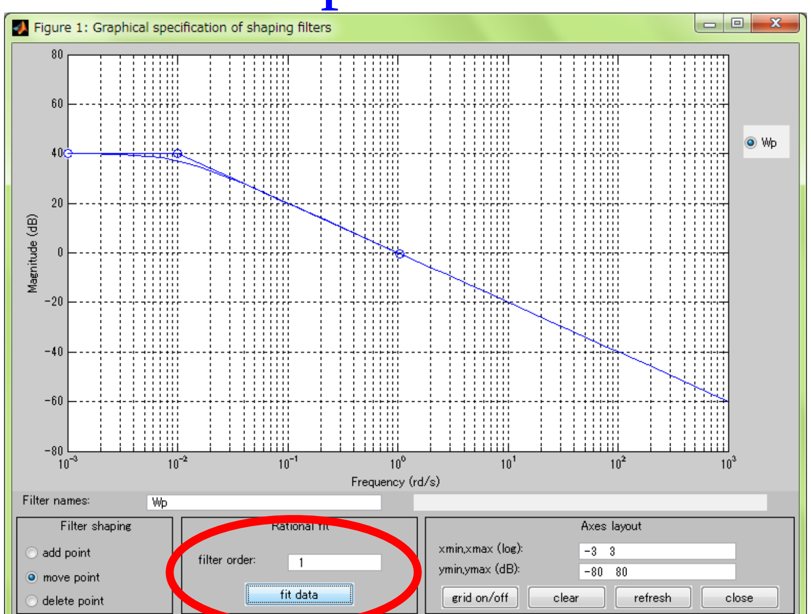
STEP 1 Input filter name

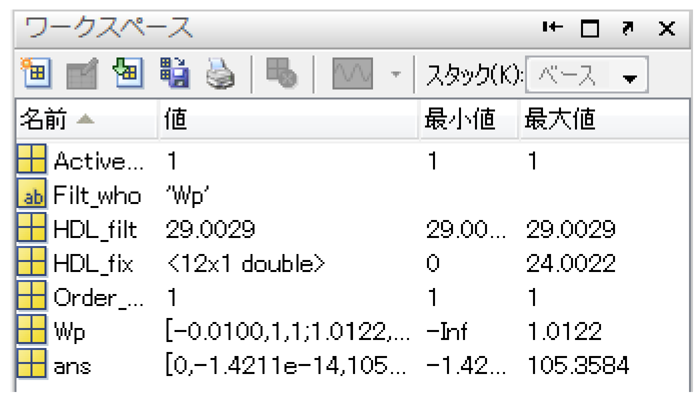


STEP 2 Put desired points



STEP 3 shape a Filter





Output the filter automatically (SYSTEM matrix form)

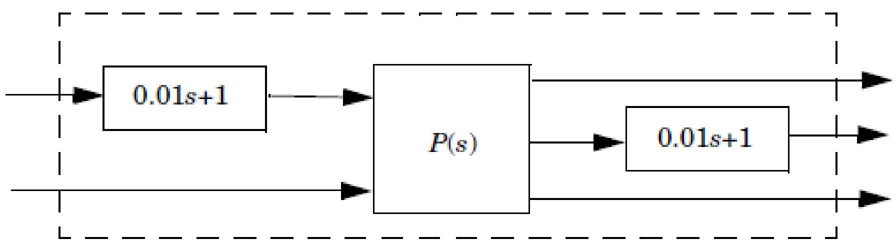
MATLAB command: "sderiv"

Alternatively, you can use "sderiv" to include nonproper shaping filters in the loop-shaping criterion. This function appends a SISO PD component to selected Input/Output of given LTI system.

Design Method

To specify more complex nonproper filters,

1. Specify the proper "low-frequency" part of the filters with magshape.
2. Augment the plant with these low-pass filters.
3. Add the derivative action of the nonproper filters by applying sderiv to the augmented plant.



is specified by

```
Pd = sderiv(P,[ 1 2],[0.01 1])
```

In the calling list, `[ 1 2]` lists the input and output channels to be filtered by $ns + d$ (here 1 for "first input" and 2 for "second output") and the vector `[0.01 1]` lists the values of n and d. An error is generated if the resulting system `Pd` is not proper.