Robust and Optimal Control, Spring 2015

Instructor: Prof. Masayuki Fujita (S5-303B)

D. Linear Matrix Inequality

D.1 Convex Optimization

D.2 Linear Matrix Inequality(LMI)

D.3 Control Design and LMI Formulation

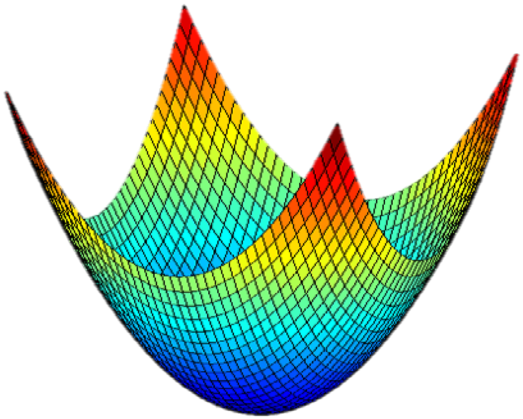
Convex Optimization Problems

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \le 0\,, & i = 1, \cdots, m \\ & Ax = b & \end{array}$$

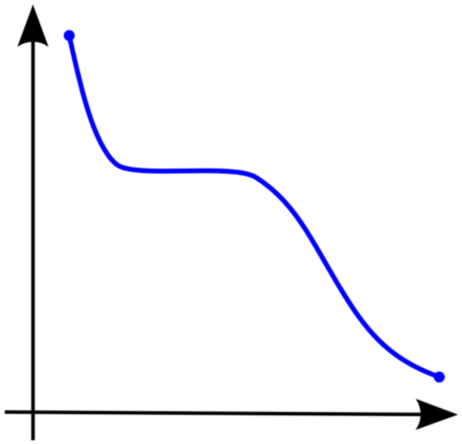
(f_0 : convex)
(f_i : convex)
(affine)

Note: A problem is quasiconvex if f_0 is quasiconvex and $f_1, \cdots, f_m$ are convex.

The feasible set of a convex (or quasiconvex) optimization problem is convex.



Convex function



Quasi-convex function

Semidefinite Programming Problem (SDP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0 \\ & Ax = b \end{array} \qquad \text{(LMI)}$$

where $F_i,\ G \in \mathbb{S}^k$

$\mathbb{S}^k$: a set of symmetric matrix (size k)

$P \preceq 0$: a symmetric matrix $P \in \mathbb{R}^{n\times n}$ is a *negative semidefinite* if the following inequality holds.

$$x^T P x \geq 0,\ \forall x \in \mathbb{R}^n$$

Note: Multiple constraints are trivially combined into a single (larger) constraint,

$$\begin{cases} x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0 \\ x_1 H_1 + x_2 H_2 + \cdots + x_n H_n + M \preceq 0 \end{cases}$$

iff $x_1 \begin{bmatrix} F_1 & 0 \\ 0 & H_1 \end{bmatrix} + x_2 \begin{bmatrix} F_2 & 0 \\ 0 & H_2 \end{bmatrix} + \cdots + x_n \begin{bmatrix} F_n & 0 \\ 0 & H_n \end{bmatrix} + \begin{bmatrix} G & 0 \\ 0 & M \end{bmatrix} \preceq 0$

Linear Matrix Inequality (LMI)

Formulation

$$F(x) := F_0 + x_1 F_1 + \cdots + x_m F_m < 0$$

$F_0,\ F_i\ (i = 1, \cdots, m) \in \mathbb{R}^{n \times n}$: Constant Symmetric Matrices

$x := [x_1, \cdots, x_m]^T \in \mathbb{R}^m$: Variables

Set of x satisfying $F(x) < 0$ is convex, i.e.,

for every x_1, x_2 satisfying $F(x_1) < 0, F(x_2) < 0$, $^\forall \alpha \in [0, 1]$

$$F(\alpha x_1 + (1 - \alpha) x_2) = \alpha F(x_1) + (1 - \alpha) F(x_2) < 0$$

➡ Convex Optimization Problem

General Formulation

$$\mathcal{G}(X_1, X_2, \cdots, X_n) < 0$$

$\mathcal{G}(\cdot)$: Symmetric Matrix, $X_i\ (i = 1, \cdots, m)$: Affine Functions

Ex. $AX + XA^T < 0$

LMI Numerical Optimization Problems

Fact Set of x satisfying LMI condition $F(x) > 0$ a convex set.

$$\left(\begin{array}{l} \text{Suppose any } x, y \text{ satisfy } F(x) > 0, F(y) > 0 \text{. Then} \\ F(\alpha x + (1-\alpha)y) \leq \alpha F(x) + (1-\alpha)F(y) < 0 \ \forall \alpha \in [0,1] \end{array} \right)$$

[Ex.] Convex Feasibility Problem(CFP)

$$\boxed{\text{find} \quad x \in \mathbb{R}^m \quad \text{s.t.} \quad F(x) > 0}$$

[Ex.] Convex Optimization Problem(COP)

$$\boxed{\min_{x \in \mathbb{R}^m} \quad c'x \qquad \text{s.t.} \quad F(x) > 0}$$

[Ex.] Quasi-convex Optimization Problem(QOP)

$$\boxed{\min_{x \in \mathbb{R}^m, \lambda > 0} \quad \lambda \qquad \text{s.t.} \quad \lambda A(x) > B(x) \text{ , } A(x) > 0 \text{ and } C(x) > 0}$$

$F(x), A(x), B(x), C(x)$: Affine functions

$c \in \mathbb{R}^m$

LMI Numerical Optimization Problems

[Ex.] Scaled H_∞ Norm Condition

For a internally stable system $G(s) = C(sI - A)^{-1}B + D$,

$$\|S^{-1/2} G S^{1/2}\|_\infty < \gamma$$

where $\gamma > 0$ and $S \in \mathcal{D}$, which is a structured sym. matrix set

[CFP] find X such that

$$F(x) := \begin{bmatrix} AX + XA' & XC' \\ CX & -\gamma^2 S \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix} S \begin{bmatrix} B \\ D \end{bmatrix}' < 0 \quad (*)$$

[COP] min $c'x = \gamma^2$ subject to (*) over X and $\gamma^2 > 0$

[Ex.] $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix}$ ➡ $x = \begin{bmatrix} \gamma^2 & x_{11} & x_{12} & x_{22} \end{bmatrix}'$, $c = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}'$

[QOP] min $\lambda := \gamma^2$ subject to (*) over X, $S \in \mathcal{D}$ and $\gamma^2 > 0$

Standard Solvers: SDP, Projection Algorithm(interior-point method)

Semidefinite Programming Problem (SDP)

[Ex.] Matrix norm minimization (Maximum singular value)

minimize $\|A(x)\|_2 = \sqrt{\rho(A(x)^T A(x))}$

where $A(x)$ is an LMI

$$A(x) = A_0 + x_1 A_1 + x_2 A_2 + \cdots + x_n A_n$$

The equivalent SDP

minimize t

subject to $\begin{bmatrix} tI & A(x) \\ A(x)^T & tI \end{bmatrix} \succeq 0$

Decision variables: t, x

$P \succeq 0$: a *positive semidefinite*

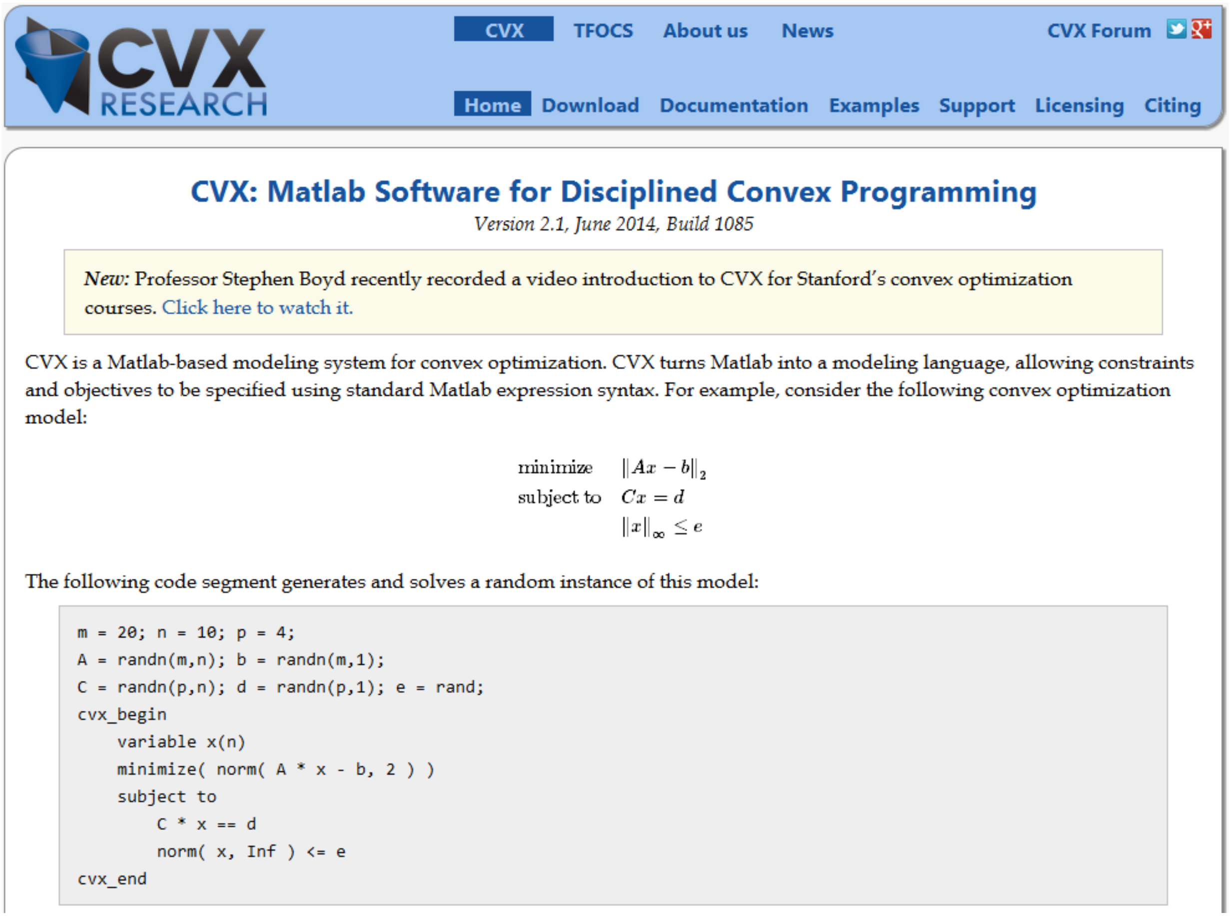
Note: The constraint equivalence follows from a Schur complement argument

$$\|A(x)\|_2 \leq t \quad \Leftrightarrow \quad A(x)^T A(x) \preceq t^2 I, \; t \geq 0$$

$$\Leftrightarrow \quad \begin{bmatrix} tI & A(x) \\ A(x)^T & tI \end{bmatrix} \succeq 0$$

LMI Programming: CVX

MATLAB Software for Disciplined Convex Programming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CVX: Matlab Software for Disciplined Convex Programming

Version 2.1, June 2014, Build 1085

New: Professor Stephen Boyd recently recorded a video introduction to CVX for Stanford's convex optimization courses. Click here to watch it.

CVX is a Matlab-based modeling system for convex optimization. CVX turns Matlab into a modeling language, allowing constraints and objectives to be specified using standard Matlab expression syntax. For example, consider the following convex optimization model:

$$\begin{array}{ll} \text{minimize} & \|Ax-b\|_2 \\ \text{subject to} & Cx = d \\ & \|x\|_\infty \le e \end{array}$$

The following code segment generates and solves a random instance of this model:

```
m = 20; n = 10; p = 4;
A = randn(m,n); b = randn(m,1);
C = randn(p,n); d = randn(p,1); e = rand;
cvx_begin
    variable x(n)
    minimize( norm( A * x - b, 2 ) )
    subject to
        C * x == d
        norm( x, Inf ) <= e
cvx_end
```

http://cvxr.com/cvx/

Michael C. Grant

Stephen P. Boyd

LMI Programming: CVX

[Ex.] Proving the stability of a system:

$$\frac{dx(t)}{dt} = Ax(t) \qquad (*)$$

CVX Command

```
cvx_begin sdp
  variable P(n,n) symmetric
  A'*P + P*A <= -eye(n)
  P >= eye(n)
cvx_end
```

The following conditions are equivalent:

(i) (*) is stable, i.e., $\mathrm{Re}[\lambda(A)] < 0 \ \forall\lambda$

(ii) $\exists P = P^T \succ 0$ s.t. $A^T P + PA \prec 0$

(iii) $\exists X = X^T \succ 0$ s.t. $XA^T + AX \prec 0$

(iv) $\exists P = P^T \succeq I$ s.t. $A^T P + PA \preceq -I$

(v) $\exists X = X^T \succeq I$ s.t. $XA^T + AX \preceq -I$

A candidate of Lyapunov function $V(x) = x(t)^T P x(t)$, $P = P^T$

Stability Condition: $\dot{V}(x) = x^T (PA + A^T P) x < 0 \ \forall x \neq 0$

Note: `cvx_status` is a string returning the status of the optimization

LMI Programming: CVX

[Ex.] Proving the stability of two systems,

$$\frac{dx(t)}{dt} = A_1 x(t) \quad \text{and} \quad \frac{dx(t)}{dt} = A_2 x(t)$$

CVX Command

```
cvx_begin sdp
  variable P(n,n) symmetric
  A1'*P + P*A1 <= -eye(n)
  A2'*P + P*A2 <= -eye(n)
  P >= eye(n)
cvx_end
```

The following conditions are equivalent:

(i) (*) is stable for $A(t) = \theta_1(t)A_1 + \theta_2(t)A_2, \;\; \theta_i(t) \geq 0$

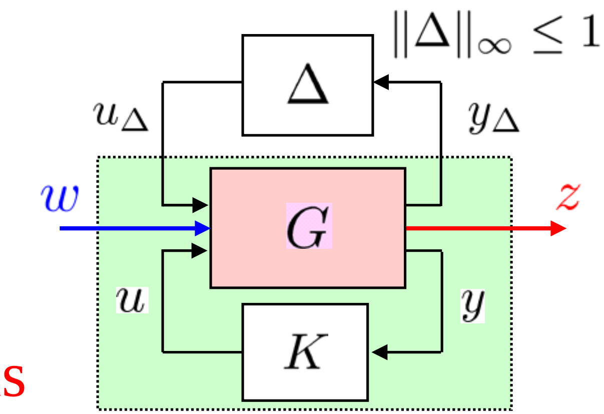
(ii) $\exists P = P^T \succ 0 \text{ s.t. } A_1^T P + PA_1 \prec 0$
and $A_2^T P + PA_2 \prec 0$

(iii) $\exists P = P^T \succeq I \text{ s.t. } A_1^T P + PA_1 \preceq -I$
and $A_2^T P + PA_2 \preceq -I$

The stability can be proven with a single Lyapunov function,

$$V(x) = x(t)^T P x(t)$$

Beyond Riccati-based H_∞ Control



Riccati-based Solution

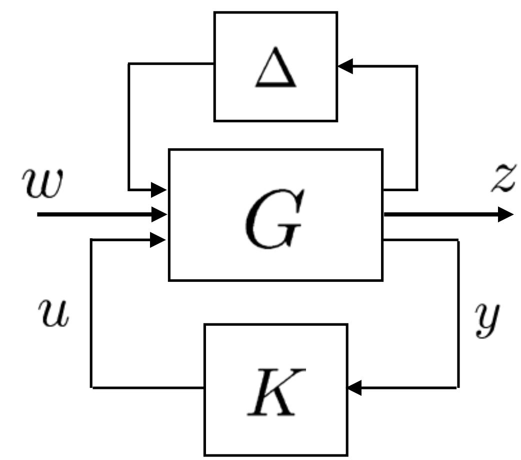
- Norm Bounded Uncertainty Type Problem
- H_∞/H_2 Control Problem (See 5th doc.)

Assumptions Full rank on the imaginary axis

$$X_\infty A + A^T X_\infty + X_\infty(\gamma^{-2}B_1B_1^T - B_2B_2^T)X_\infty + C_1^TC_1 = 0$$
$$AY_\infty + Y_\infty A^T + Y_\infty(\gamma^{-2}C_1^TC_1 - C_2^TC_2)Y_\infty + B_1B_1^T = 0$$

Equalities

LMI-based Solution



- Matrix Polytope Type Control Problem
- Singular Matrix Type Control Problem
- Quadratic Stabilization Problem
- Gain Scheduled Control Problem
- Multi-objective Control Problem

There is NO assumption about general plants

$$\begin{bmatrix} A^TX + XA + BB^T & XC^T + BD^T \\ CX + DB^T & DD^T - \gamma^2 I \end{bmatrix} < 0$$

Inequalities

Robust Stability Condition

Stable LTI system $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$

$$G(s) = C(sI - A)^{-1}B + D$$

Given $\gamma > 0$, the following conditions are equivalent.

(i) Stability of H_∞-norm A is stable and $\|G\|_\infty < \gamma$

(ii) Algebraic Riccati Equation

$$^{\exists}X = X^T \geq 0, \;\; R := \gamma^2 I - DD^T > 0$$

$$AX + XA^T + (XC^T + BD^T)R^{-1}(XC^T + BD^T)^T + BB^T = 0$$

(iii) Riccati Inequality $^{\exists}X = X^T > 0, \;\; R := \gamma^2 I - DD^T > 0$

$$AX + XA^T + (XC^T + BD^T)R^{-1}(XC^T + BD^T)^T + BB^T < 0$$

(iv) LMI $^{\exists}X = X^T > 0, \begin{bmatrix} A^T X + XA + BB^T & XC^T + BD^T \\ CX + DB^T & DD^T - \gamma^2 I \end{bmatrix} < 0$

Robust Stability Condition (Cont'd)

Stable LTI system $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$

$$G(s) = C(sI - A)^{-1}B + D$$

Given $\gamma > 0$, the following conditions are equivalent.

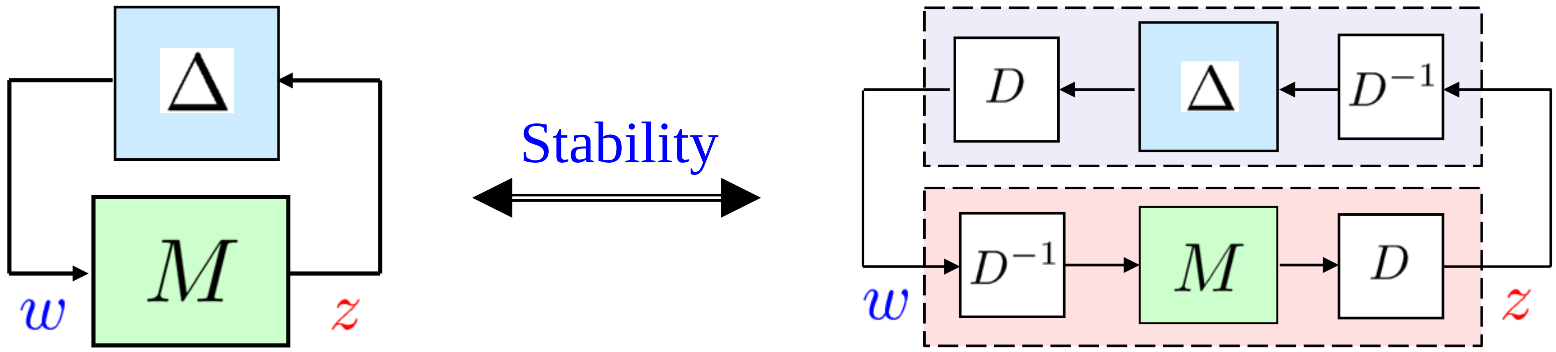
(iv) LMI $\exists X = X^T > 0, \begin{bmatrix} A^TX + XA + BB^T & XC^T + BD^T \\ CX + DB^T & DD^T - \gamma^2 I \end{bmatrix} < 0$

(v) LMI $\exists X = X^T > 0, \begin{bmatrix} AX + XA^T & XC^T & B \\ CX & -\gamma I & D \\ B^T & D^T & -\gamma I \end{bmatrix} < 0$

(v)' LMI $\exists P = P^T > 0, \begin{bmatrix} A^TP + PA & PB & C^T \\ B^TP & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0$

 Schur Complement

LMI formulation: Structured Singular Value [SP05, p. 478]



Upper Bound $\mu_\Delta(M) \le \min_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1}) < \gamma$

$$\Leftrightarrow \gamma^2 I - (DMD^{-1})^H (DMD^{-1}) \succ 0$$

$$\Leftrightarrow \gamma^2 I - D^{-1} M^H D^2 M D^{-1} \succ 0$$

$$\Leftrightarrow \gamma^2 D^2 - M^H D^2 M \succ 0 \Leftrightarrow \gamma^2 D - M^H D M \succ 0$$

If γ varies monotomically, the feasible regions of $D \in \mathcal{D}$ are nested

$$\begin{array}{ll} \underset{\eta,\, D \in \mathcal{D}}{\text{minimize}} & \eta \\ \text{subject to} & \eta D - M^H D M \succ 0 \end{array}$$

Quasiconvex optimization problem:
Generalized eigenvalue problem

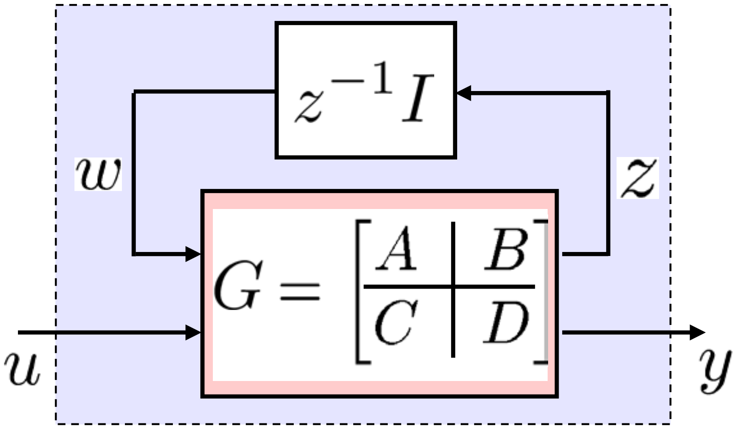
Then $\gamma = \sqrt{\eta^{\text{opt}}}$ is an upper bound for $\mu_\Delta(M)$

LMI formulation: via Main Loop Theorem

State-space performance test

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \qquad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

$$y = F_u(G, z^{-1}I)u$$



$$\mu_\Delta(G) < 1 \quad \Leftrightarrow \quad \begin{cases} F_u(G, z^{-1}I) \ \text{ is stable} \\ \|F_u(G, z^{-1}I)\|_\infty < 1 \end{cases}$$

$$\Delta = \{\text{diag}(\delta_1 I_{nx}, \Delta_2) | \delta_1 \in \mathcal{C}, \Delta_2 \in \mathcal{C}^{nu \times ny}\}$$

In this case $\mu_\Delta(G) = \inf_{D \in \mathcal{D}} \bar{\sigma}(DGD^{-1})$

$$\mathcal{D} = \left\{ \begin{bmatrix} D_1 & 0 \\ 0 & d_2 I \end{bmatrix} \middle| D_1 = D_1^H \succ 0, d_2 > 0 \right\}$$

Consider (without loss of generality) finding D_1 such that

$$\bar{\sigma}\left(\begin{bmatrix} D_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} D_1^{-1} & 0 \\ 0 & I \end{bmatrix} \right) < 1$$

LMI formulation: Bounded Real Lemma

State-space performance test

$$\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^T - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \right) \prec 0$$

$$X = X^H \succ 0 \qquad \text{(take } X = D_1^2 \text{)}$$

Bounded Real Lemma

$$\begin{cases} F_u(G, z^{-1}I) \text{ is stable} \\ \|F_u(G, z^{-1}I)\|_{\mathcal{L}_2} < 1 \end{cases}$$

$$\Leftrightarrow \; \exists X = X^H \succ 0 \quad \text{s.t.} \begin{bmatrix} -X & 0 & A^T X & C^T \\ 0 & -I & B^T X & D^T \\ XA & XB & -X & 0 \\ C & D & 0 & -I \end{bmatrix} \prec 0$$

LMI formulation: Bounded Real Lemma

Discrete-time

$$\exists Y = Y^H \succ 0 \quad \text{s.t.}$$

$$\begin{cases} F_u(G, z^{-1}I) \ \text{ is stable} \\ \|F_u(G, z^{-1}I)\|_\infty < \gamma \end{cases} \Leftrightarrow \begin{bmatrix} Y & AY & B & 0 \\ YA^T & Y & 0 & YC^T \\ B^T & 0 & I & D^T \\ 0 & CY & D & \gamma^2 I \end{bmatrix} \succ 0$$

$$\begin{bmatrix} Y & AY \\ YA^T & Y \end{bmatrix} \succ 0 \Leftrightarrow AYA^T - Y \prec 0$$

Discrete-time Lyapunov condition

Continuous-time

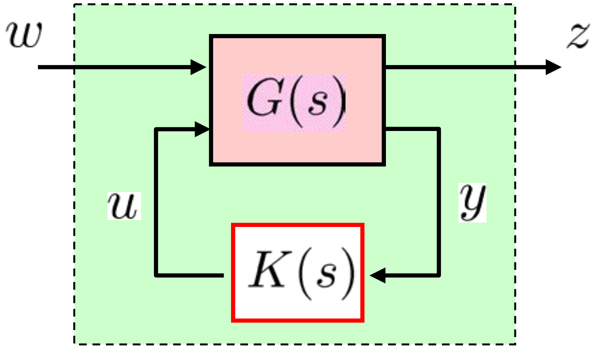
$$\exists P = P^H \succ 0$$

$$\begin{cases} F_u(G, z^{-1}I) \ \text{ is stable} \\ \|F_u(G, z^{-1}I)\|_\infty < \gamma \end{cases} \Leftrightarrow \begin{bmatrix} A^TP + PA & PB & C^T \\ B^TP & -I & D^T \\ C & D & -\gamma^2 I \end{bmatrix} \prec 0$$

$$\exists Q = P^{-1}$$

$$\Leftrightarrow \begin{bmatrix} QA^T + AQ & B & QC^T \\ B^T & -I & D^T \\ CQ & D & -\gamma^2 I \end{bmatrix} \prec 0$$

State feedback H_∞ control



$$G = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ I & 0 & 0 \end{array}\right] \qquad (A, B_u) : \text{stabilizable}$$

$$\begin{bmatrix} z \\ y \end{bmatrix} = G(s) \begin{bmatrix} w \\ u \end{bmatrix} \quad \text{and} \quad u = Kx = Ky$$

$$F_l(G, K) = \left[\begin{array}{c|c} A + B_u K & B_w \\ \hline C_z + D_{zu} K & D_{zw} \end{array}\right]$$

Continuous-time $F = KQ$, $\eta = \gamma^2$

$$\begin{aligned} &\underset{\eta, Q, F}{\text{minimize}} \quad \eta \\ &\text{subject to} \quad Q = Q^T \prec 0 \\ &\begin{bmatrix} QA^T + F^T B_u^T + AQ + B_u F & B_w & QC_z^T + F^T D_{zu}^T \\ B_w^T & -I & D_{zw}^T \\ C_z Q + D_{zu} F & D_{zw} & -\eta I \end{bmatrix} \prec 0 \end{aligned}$$

$K = FQ^{-1}$ gives $F_l(G, K)$ stable and $\|F_l(G, K)\|_\infty \leq \sqrt{\eta}$

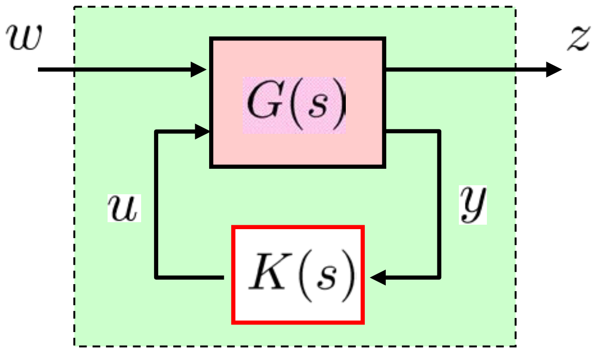
State feedback H_∞ control: CVX

$$\begin{aligned}
&\underset{\eta, Q, F}{\text{minimize}} \quad \eta \\
&\text{subject to} \quad Q = Q^T \prec 0 \\
&\begin{bmatrix} QA^T + F^T B_u^T + AQ + B_u F & B_w & QC_z^T + F^T D_{zu}^T \\ B_w^T & -I & D_{zw}^T \\ C_z Q + D_{zu} F & D_{zw} & -\eta I \end{bmatrix} \prec 0
\end{aligned}$$

CVX Command

```
P = ss(A, [Bw, Bu], [Cz; eye(n,n)], [Dzw, Dzu; zeros(n,nw+nu)]) ;
cvx_begin sdp
  variable Q(n,n) symmetric;
  variable F(nu,n);
  variable eta;
  minimize eta ;
  subject to:
    Q > 0 ;
    [ Q*A' + F'*Bu' + A*Q + Bu*F, Bw,              Q*Ce' + F'*Dzu' ;
      Bw',                        -eye(nw,nw),  Dzw' ;
      Cz*Q + Dzu*F,               Dzw,          -eta*eye(nz,nz)] < 0 ;
cvx_end
K = F*inv(Q) ;
Aclp = A + Bu*K ;
Disp( eig(Aclp) ) ;  % always check that it really is a good controller.
```

Output feedback H_∞ control



$$G = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right]$$

(A, B_u) : stabilizable

(C_y, A) : detectable

$$\begin{bmatrix} z \\ y \end{bmatrix} = G(s) \begin{bmatrix} w \\ u \end{bmatrix} \quad \text{and} \quad u = K(s)y = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & 0 \end{array}\right] y$$

$$F_l(G,K) = \left[\begin{array}{cc|c} A & B_u C_k & B_w \\ B_k C_y & A_k & B_k D_{yw} \\ \hline C_z & D_{zu} C_k & D_{zw} \end{array}\right] = \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right]$$

Continuous-time

$$\begin{cases} F_u(G,K) \ \text{is stable} \\ \|F_u(G,K)\|_\infty < \gamma \end{cases} \Leftrightarrow \quad \exists P = P^H \succ 0 \quad \begin{bmatrix} A_{cl}^T P + P A_{cl} & P B_{cl} & C_{cl}^T \\ B_{cl}^T P & -I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma^2 I \end{bmatrix} \prec 0$$

Output feedback H_∞ control

Partition P as:

$$P = \begin{bmatrix} Y & N \\ N^T & * \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} X & M \\ M^T & * \end{bmatrix}$$

$$\hat{A} = NA_kM^T + NB_kC_yX + YB_uC_kM^T + YAX$$
$$\hat{B} = NB_k$$
$$\hat{C} = C_kM^T$$

Define an inertia-preserving transform via: $T = \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}$

$$\begin{aligned}
T^TPA_{cl}T &= \begin{bmatrix} AX + B_u\hat{C} & A \\ \hat{A} & YA + \hat{B}C_y \end{bmatrix} \\
T^TPB_{cl} &= \begin{bmatrix} B_w \\ YB_w + \hat{B}D_{yw} \end{bmatrix} \\
C_{cl}T &= \begin{bmatrix} C_zX + D_{zu}\hat{C} & C_z \end{bmatrix} \\
T^TPT &= \begin{bmatrix} X & I \\ I & Y \end{bmatrix}
\end{aligned}$$

Output feedback H_∞ control

$$\begin{aligned}
&\underset{\eta, X, Y, \hat{A}, \hat{B}, \hat{C}}{\text{minimize}} \quad \eta \\
&\text{subject to} \quad \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succ 0 \\
&\begin{bmatrix} T^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{cl}^T P + P A_{cl} & P B_{cl} & C_{cl}^T \\ B_{cl}^T P & -I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma^2 I \end{bmatrix} \begin{bmatrix} T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \prec 0
\end{aligned}$$

$$\Leftrightarrow \begin{bmatrix} AX + B_u\hat{C} + XA^T + \hat{C}^T B_u^T & A + \hat{A}^T & B_w & XC_z^T + \hat{C}^T D_{zu}^T \\ A^T + \hat{A} & YA + A^T Y + \hat{B}C_y + C_y^T \hat{B}^T & YB_w + \hat{B}D_{yw} & C_z^T \\ B_w^T & B_w^T Y + D_{yw}^T \hat{B}^T & -I & D_{zw}^T \\ C_z X + D_{zu}\hat{C} & C_z & D_{zw} & -\eta I \end{bmatrix} \prec 0$$

$$PP^{-1} = I \quad \Rightarrow \quad NM^T = I - YX$$

$$K(s) = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & 0 \end{array}\right]$$ gives $F_l(G,K)$ stable and $\|F_l(G,K)\|_\infty \leq \sqrt{\eta}$

Output feedback H_∞ control: CVX

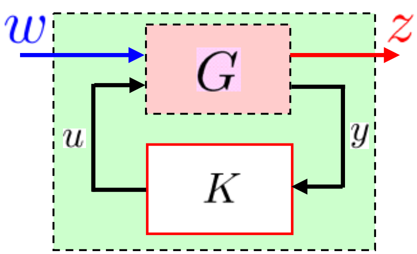
CVX Command

```
P = ss(A, [Bw, Bu], [Cz; Cy], [Dzw, Dzu; Dyw, zeros(ny,nu)]) ;

cvx_begin sdp
  variable X(n,n) symmetric;
  variable Y(n,n) symmetric;
  variable Ah(n,n);
  variable Bh(n,ny);
  variable Ch(nu,n);
  variable eta;

  minimize eta ;
  subject to:
    [ X,           eye(n,n) ;
      eye(n,n), Y            ] > 0 ;
    [ A*X + Bu*Ch + X*A' + Ch'*Bu', A+Ah', Bw, X*Ce' + Ch'*Dzu' ;
      A'+Ah,                          Y*A + A'*Y + Bh*Cy + Cy'*Bh', Y*Bw + Bh*Dyw, Ce' ;
      Bw',                            Bw'+Y + Dyw'*Bh',              -eye(nw,nw),     Dzw' ;
      Cz*X + Dzu*Ch,                  Cz,                            Dzw,    -eta*eye(nz,nz)] < 0 ;
cvx_end
```

Same as **hinfsyn**

MATLAB Command

```
[Khi,CLhi,ghi,hiinfo] = hinfsyn(P,ny,nu,'Method','lmi');
```



$$z = F_l(G, K)w$$

H_2 control

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \qquad \|G(s)\|_{\mathcal{L}_2} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Tr}(G(j\omega)G^*(j\omega))d\omega}$$

Theorem

$$\left\{\begin{array}{l} G(s) \quad \text{is stable} \\ \|G(s)\|_{\mathcal{H}_2}^2 < \gamma \end{array}\right. \Leftrightarrow \quad \exists X = X^T \succ 0 \;\text{ s.t.}\; \left\{\begin{array}{l} \mathrm{Tr}(CXC^T) < \gamma \\ AX + XA^T + BB^T \prec 0 \end{array}\right.$$

Continuous-time

$\exists X = X^T \succ 0$ s.t.

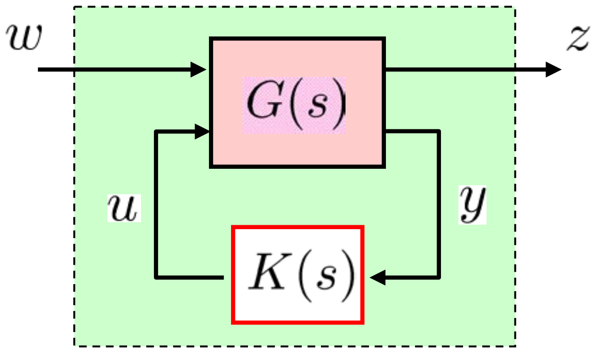
$$\left\{\begin{array}{l} AX + XA^T + BB^T \prec 0 \\ \begin{bmatrix} W & CX \\ XC^T & X \end{bmatrix} \succ 0 \\ \mathrm{Tr}(W) < \gamma \end{array}\right.$$

Discrete-time

$\exists X = X^T \succ 0$ s.t.

$$\left\{\begin{array}{l} \begin{bmatrix} X & AX & B \\ XA^T & X & 0 \\ B^T & 0 & I \end{bmatrix} \succ 0 \\ \begin{bmatrix} W & CX \\ XC^T & X \end{bmatrix} \succ 0 \\ \mathrm{Tr}(W) < \gamma \end{array}\right.$$

State feedback H_2 control



$$G = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & 0 & D_{zu} \\ I & 0 & 0 \end{array}\right] \qquad (A, B_u) : \text{stabilizable}$$

$$\begin{bmatrix} z \\ y \end{bmatrix} = G(s) \begin{bmatrix} w \\ u \end{bmatrix} \quad \text{and} \quad u = Kx = Ky$$

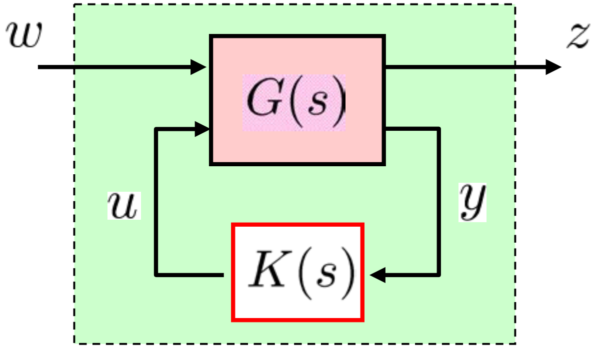
$$F_l(G, K) = \left[\begin{array}{c|c} A + B_u K & B_w \\ \hline C_z + D_{zu} K & 0 \end{array}\right]$$

Continuous-time $G(s)$ is stable and $\|G(s)\|_{\mathcal{L}_2} < \gamma$ iff

$\exists X = X^T \succ 0$ and $F = KX$ s.t.

$$\begin{cases} AX + B_u F + XA^T + F^T B_u^T + B_w B_w^T \prec 0 \\ \begin{bmatrix} W & C_z X + D_{zu} F \\ X C_z^T + F^T D_{zu}^T & X \end{bmatrix} \succ 0 \\ \mathrm{Tr}(W) < \gamma \end{cases}$$

H_2 control: LQG Problem



$$G = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & 0 & D_{zu} \\ I & 0 & 0 \end{array}\right] \qquad (A, B_u) : \text{stabilizable}$$

$$\begin{bmatrix} z \\ y \end{bmatrix} = G(s) \begin{bmatrix} w \\ u \end{bmatrix} \quad \text{and} \quad u = Kx = Ky$$

LQG Objective: $J = \sum_{k=0}^{\infty} x(k)^T Q x(k) + u(k)^T R u(k)$

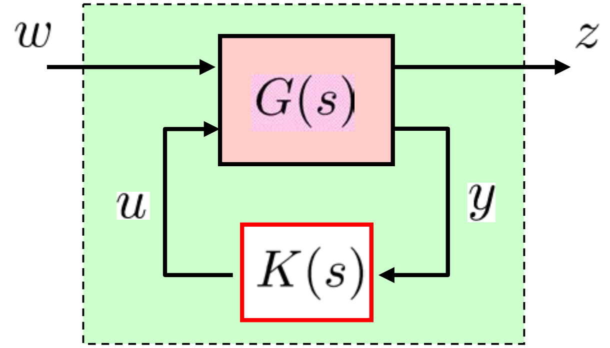
$$\Rightarrow \quad C_z = \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} \quad \text{and} \quad D_{zu} = \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix}$$

$$\Rightarrow \quad e(k) = \begin{bmatrix} Q^{1/2} x(k) \\ R^{1/2} u(k) \end{bmatrix}$$

$$e(k)^T e(k) = x(k)^T Q x(k) + u(k)^T R u(k)$$

$$\Rightarrow \quad \|e(k)\|_2^2 = \sum_{k=0}^{\infty} x(k)^T Q x(k) + u(k)^T R u(k)$$

Output feedback H_2 control



$$G = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & 0 & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right]$$

(A, B_u) : stabilizable

(C_y, A) : detectable

$$\begin{bmatrix} z \\ y \end{bmatrix} = G(s) \begin{bmatrix} w \\ u \end{bmatrix} \quad \text{and} \quad u = K(s)y = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & 0 \end{array}\right] y$$

$$F_l(G, K) = \left[\begin{array}{cc|c} A & B_u C_k & B_w \\ B_k C_y & A_k & B_k D_{yw} \\ \hline C_z & D_{zu} C_k & 0 \end{array}\right] = \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & 0 \end{array}\right]$$

Continuous-time $G(s)$ is stable and $\|G(s)\|_{\mathcal{L}_2} < \gamma$ iff

$\exists P = P^H \succ 0$ s.t.

$$\begin{bmatrix} A_{cl}^T P + P A_{cl} & P B_{cl} \\ B_{cl}^T P & -I \end{bmatrix} \prec 0 \, , \quad \begin{bmatrix} W & C_{cl} \\ C_{cl}^T & P \end{bmatrix} \succ 0 \ \text{ and } \ P \succ 0$$

Output feedback H_2 control

$$\underset{\gamma, W, X, Y, \hat{A}, \hat{B}, \hat{C}}{\text{minimize}} \quad \gamma$$

$$\text{subject to} \quad \mathrm{Tr}(W) < \gamma$$

$$\begin{bmatrix} T^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{cl}^T P + P A_{cl} & P B_{cl} \\ B_{cl}^T P & -I \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \prec 0$$

$$\begin{bmatrix} I & 0 \\ 0 & T^T \end{bmatrix} \begin{bmatrix} W & C_{cl} \\ C_{cl}^T & P \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix} \succ 0$$

$$\Leftrightarrow \begin{bmatrix} AX + B_u\hat{C} + XA^T + \hat{C}^T B_u^T & A + \hat{A}^T & B_w \\ A^T + \hat{A} & YA + A^T Y + \hat{B} C_y + C_y^T \hat{B}^T & YB_w + \hat{B} D_{yw} \\ B_w^T & B_w^T Y + D_{yw}^T \hat{B}^T & -I \end{bmatrix} \prec 0$$

$$\begin{bmatrix} W & C_z X + D_{zu}\hat{C} \\ XC_z^T + \hat{C}^T D_{zu}^T & X \end{bmatrix} \succ 0$$

$$PP^{-1} = I \quad \Rightarrow \quad NM^T = I - YX$$

$K(s) = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & 0 \end{array}\right]$ gives $F_l(G,K)$ stable and $\|F_l(G,K)\|_{\mathcal{H}_2} \leq \sqrt{\eta}$

H_2 control: CVX

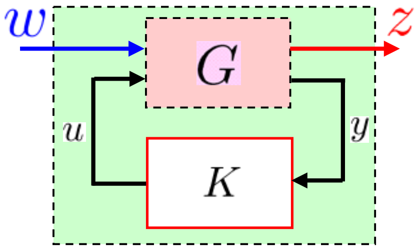
CVX Command

```
cvx_begin sdp
  variable X(n,n) symmetric;
  variable Y(n,n) symmetric;
  variable W(nz,nz) symmetric;
  variable Ah(n,n);
  variable Bh(n,ny);
  variable Ch(nu,n);
  variable gamma;

  minimize gamma ;
  subject to:
    trace(W) < gamma ;
    [ W,               Cz*X+Dzu*Ch,  Cz ;
      X*Cz’+Ch’*Dzu’, X,             eye(n,n) ;
      Cz’,             eye(n,n),      Y        ] > 0 ;
    [ A*X + Bu*Ch + X*A’ + Ch’*Bu’, A+Ah’,                         Bw ;
      A’+Ah,                        Y*A + A’*Y + Bh*Cy + Cy’*Bh’, Y*Bw + Bh*Dyw ;
      Bw’,                          Bw’+Y + Dyw’*Bh’,              -eye(nw,nw) ] < 0 ;
cvx_end
```

Same as **h2syn**

MATLAB Command

```
[K2,CL2,g2,hiinfo] = hinfsyn(P,ny,nu);
```



$$z = F_l(G, K)w$$

l_1 Design Problem

Bounding error amplitudes for bounded amplitude inputs

$$\|M\|_\infty = \sup_{\|x\|_\infty \le 1} \|Mx\|_\infty = \max_{1 \le i \le p} \sum_{j=1}^{q} |a_{ij}|$$

$$y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \end{bmatrix} = Mu = \begin{bmatrix} m_1 & 0 & 0 & \cdots \\ m_2 & m_1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \end{bmatrix}$$

Use impulse response matrices and a Youla parametrization to set up the design problem:

$$\min_Q \|P + UQV\|_\infty$$

Robust problems can also be set up and solved as (large) optimization problems

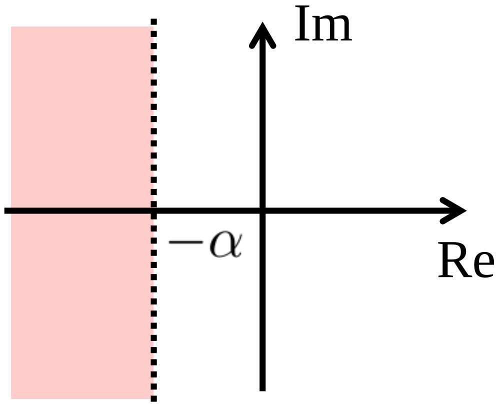
Pole Region Constraints($\mathcal{D}$ -Stability)

Definitions $f_{\mathcal{D}}(z) : \mathcal{C} \to \mathcal{S}^{p\times p} \quad f_{\mathcal{D}}(z) = L + zM + z^* M^T$

$\mathcal{D} = \{z \in \mathcal{C} | f_{\mathcal{D}}(z) \prec 0\}$: a region of the complex plane

$L = L^T \in \mathbb{R}^{p\times p}$, $M \in \mathbb{R}^{p\times p}$

[Ex.] $\mathrm{Re}(z) < -\alpha$

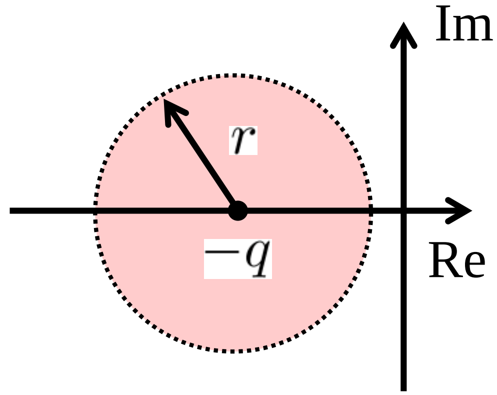


$$f_{\mathcal{D}}(z) = 2\alpha + z + z^*$$

$$L = 2\alpha$$

$$M = 1$$

[Ex.] $|z+q| < r$

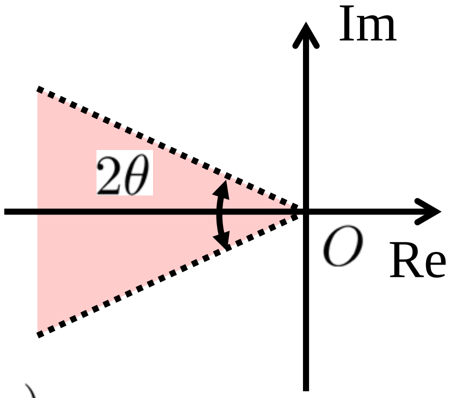


$$f_{\mathcal{D}}(z) = \begin{bmatrix} -r & q+z \\ q+z^* & -r \end{bmatrix}$$

$$L = \begin{bmatrix} -r & q \\ q & -r \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

[Ex.] Conic sector



$$f_{\mathcal{D}}(z) = \begin{bmatrix} -(z+z^*)\sin\theta & (z-z^*)\cos\theta \\ (z^*-z)\cos\theta & (z+z^*)\sin\theta \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$$

Pole Region Constraints: LMI conditions

Definitions $M_{\mathcal{D}}(A,P) = L \otimes P + M \otimes (AP) + M^T \otimes (PA^T)$

$$A \in \mathbb{R}^{n \times n}, \quad P = P^T \in \mathbb{R}^{n \times n}$$

Theorem $\mathrm{eig}(A) \in \mathcal{D}$

$$\Leftrightarrow \quad \exists P = P^H \succ 0 \ \text{ s.t. } \ M_{\mathcal{D}}(A,P) \prec 0$$

[Ex.] All closed loop poles have real part less than $-\alpha$

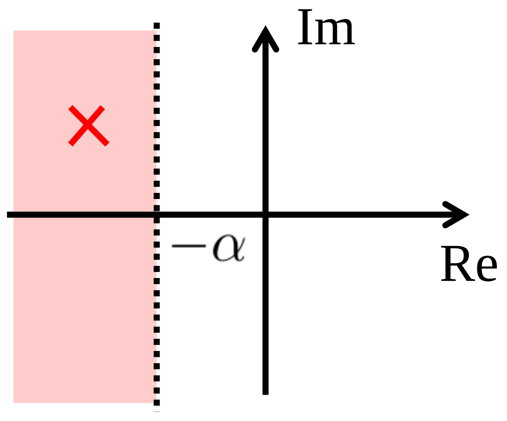
$$f_{\mathcal{D}}(z) = 2\alpha + z + z^*$$

$$L = 2\alpha \qquad M = 1$$

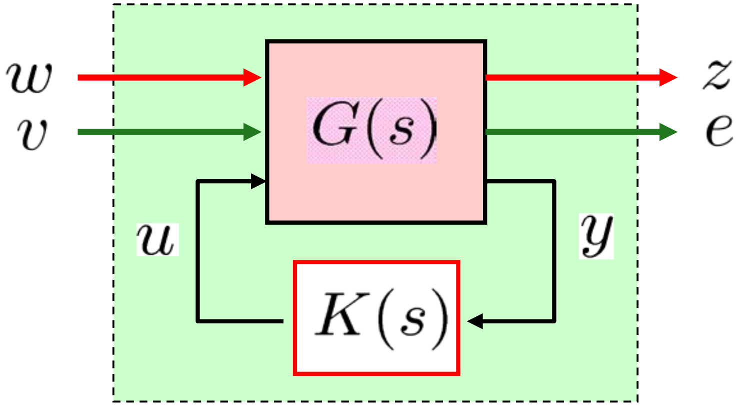
$$\mathrm{eig}(A_{cl}) \in \mathcal{D}$$

$$\Leftrightarrow \quad \exists P = P^H \succ 0 \ \text{ s.t. }$$

$$M_{\mathcal{D}}(A_{cl}, P) = 2\alpha P + A_{cl}P + PA_{cl}^T \prec 0$$



Multi-objective Analysis



$$F_l(G,K) = \left[\begin{array}{c|cc} A & B_w & B_v \\ \hline C_z & D_{zw} & D_{zv} \\ C_e & D_{ew} & 0 \end{array}\right]$$

H_∞ Control Problem $\quad w \rightarrow z$

$$\left\| \begin{bmatrix} I & 0 \end{bmatrix} F_l(G,K) \begin{bmatrix} I \\ 0 \end{bmatrix} \right\|_{\mathcal{L}_\infty} \le \gamma \quad \Leftrightarrow \quad \exists P_1 = P_1^T \succ 0 \text{ s.t. } \begin{bmatrix} A^T P_1 + P_1 A & P_1 B_w & C_z^T \\ B_w^T P_1 & -I & D_{zw}^T \\ C_z & D_{zw} & -\gamma^2 I \end{bmatrix} \prec 0$$

H_2 Control Problem $\quad v \rightarrow e$

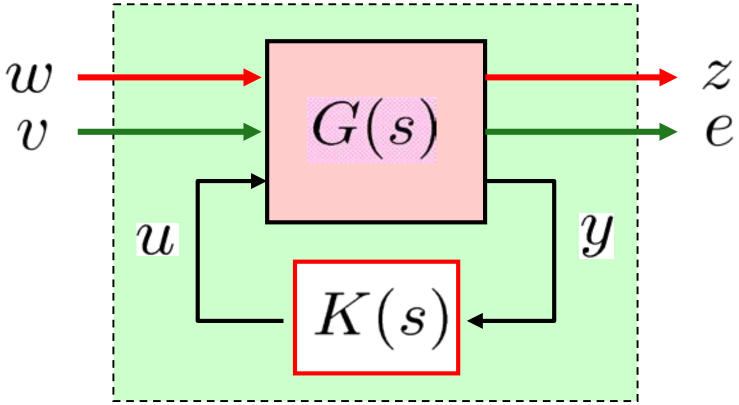
$$\left\| \begin{bmatrix} 0 & I \end{bmatrix} F_l(G,K) \begin{bmatrix} 0 \\ I \end{bmatrix} \right\|_{\mathcal{L}_2} \le \beta \quad \Leftrightarrow \quad \exists P_2 = P_2^T \succ 0 \text{ s.t. } \begin{cases} A P_2 + P_2 A^T + B_v B_v^T \prec 0 \\ \begin{bmatrix} W & C_e P_2 \\ P_2 C_e^T & P_2 \end{bmatrix} \succ 0 \quad \mathrm{Tr}(W) < \beta \end{cases}$$

Pole Region Constraints

$$\mathrm{Re}(\mathrm{eig}(A)) < -\alpha \quad \Leftrightarrow \quad \exists P_3 = P_3^T \succ 0 \text{ s.t. } 2\alpha P_3 + A P_3 + P_3 A^T \prec 0$$

Multi-objective Design

$$F_l(G,K) = \left[\begin{array}{c|cc} A & B_w & B_v \\ \hline C_z & D_{zw} & D_{zv} \\ C_e & D_{ew} & 0 \end{array}\right]$$



For Synthesis $P = P_1 = P_2 = P_3$

H_∞ Control Problem $w \to z$

$$\left\| \begin{bmatrix} I & 0 \end{bmatrix} F_l(G,K) \begin{bmatrix} I \\ 0 \end{bmatrix} \right\|_{\mathcal{L}_\infty} \le \gamma \iff \exists P = P^T \succ 0 \text{ s.t. } \begin{bmatrix} A^TP + PA & PB_w & C_z^T \\ B_w^TP & -I & D_{zw}^T \\ C_z & D_{zw} & -\gamma^2 I \end{bmatrix} \prec 0$$

H_2 Control Problem $v \to e$

$$\left\| \begin{bmatrix} 0 & I \end{bmatrix} F_l(G,K) \begin{bmatrix} 0 \\ I \end{bmatrix} \right\|_{\mathcal{L}_2} \le \beta \iff \exists P = P^T \succ 0 \text{ s.t. } \begin{cases} AP + PA^T + B_vB_v^T \prec 0 \\ \begin{bmatrix} W & C_eP \\ PC_e^T & P \end{bmatrix} \succ 0 \quad \mathrm{Tr}(W) < \beta \end{cases}$$

Pole Region Constraints

$$\mathrm{Re}(\mathrm{eig}(A)) < -\alpha \iff \exists P = P^T \succ 0 \text{ s.t. } 2\alpha P + AP + PA^T \prec 0$$

Conservative Design

Schur Complement

The following conditions are equivalent:

(i) $\begin{bmatrix} P & S \\ S^T & Q \end{bmatrix} < 0$

(ii) $P < 0, \; Q - S^T P^{-1} S < 0$

(iii) $Q < 0, \; P - S Q^{-1} S^T < 0$

(i) ⇔ (ii)

$$\begin{bmatrix} I & 0 \\ -S^T P^{-1} & I \end{bmatrix} \begin{bmatrix} P & S \\ S^T & Q \end{bmatrix} \begin{bmatrix} I & -P^{-1}S \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} P & 0 \\ 0 & Q - S^T P^{-1} S \end{bmatrix}$$

(i) ⇔ (iii)

$$\begin{bmatrix} I & -SQ^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} P & S \\ S^T & Q \end{bmatrix} \begin{bmatrix} I & 0 \\ -Q^{-1}S^T & I \end{bmatrix}$$

$$= \begin{bmatrix} P - SQ^{-1}S^T & 0 \\ 0 & Q \end{bmatrix}$$

LMI Formulation: Root mean square(RMS) Gain

RMS gain of the stable LTI system (A, B, C, D) is the minimum value of the solution γ satisfying the following statement.

$$\exists P = P^T > 0 \text{ and } \gamma \text{ s.t. } \begin{bmatrix} A^TP + PA & PB & C^T \\ B^TP & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0$$

RMS gain is the value that the average size for the sustainable signal

$$\|w\|_{\text{RMS}} := \left(\lim_{T\to\infty} \frac{1}{T} \int_0^T \|w(t)\|^2 dt \right)^{1/2}$$

$$= \text{Tr}[R_w(0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Tr}[S_w(j\omega)] d\omega$$

Covariance matrix $R_w(\tau) := \lim_{T\to\infty} \frac{1}{T} \int_0^T w(t)w(t+\tau)^T dt$

Power spectrum density $S_w(j\omega) := \int_{-\infty}^{\infty} R_w(\tau) e^{-j\omega\tau} d\tau$

If $w(t)$ satisfies ergodicity, i.e., a stationary stochastic signal,

$$\|w\|_{\text{RMS}} = E[\|w(t)\|^2]$$

Relaxations for Structured Uncertainty

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \qquad \begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

Fundamental Stability (NS)

$\exists P = P^T \succ 0$ s.t.

$$\begin{bmatrix} P & AP \\ PA^T & P \end{bmatrix} \succ 0$$

$\Leftrightarrow$

$\exists P = P^T \succ 0$ and G s.t.

$$\begin{bmatrix} P & AG \\ GA^T & G + G^T - P \end{bmatrix} \succ 0$$

H_∞ Stability

$\exists P = P^T \succ 0$ and G s.t.

$$\begin{bmatrix} P & AG & B & 0 \\ G^T A^T & G + G^T - P & 0 & G^T C^T \\ B^T & 0 & I & D^T \\ 0 & CG & D & \gamma^2 I \end{bmatrix} \succ 0$$

H_2 (Quadratic) Stability

$\exists P = P^T \succ 0$ and G s.t.

$$\begin{cases} \begin{bmatrix} P & AG & B \\ G^T A^T & G^T + G - P & 0 \\ B^T & 0 & I \end{bmatrix} \succ 0 \\ \begin{bmatrix} W & CG \\ G^T C^T & G^T + G - P \end{bmatrix} \succ 0 \\ \mathrm{Tr}(W) < \gamma \end{cases}$$

Strictly Bounded Real Lemma

Suppose $G(s) = C(sI - A)^{-1}B + D$.

Then the following are equivalent conditions.

(i) Stability of H_∞-norm

The matrix A is Hurwitz (stable) and $\|G(s)\|_\infty < 1$

A. Hurwitz

(ii) LMI

There exists a symmetric matrix $X = X^T > 0$ such that

$$\begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} A^T X + XA & XB \\ B^T X & -I \end{bmatrix} < 0$$

(iii) KYP Lemma

$I - D^T D > 0$ and there exist symmetric matrices P, Q and matrices L, W such that

R.E.Kalman L. Yakubovich

$$PA + A^T P = -C^T C - Q - L^T L$$

$$PB = -C^T D - L^T W$$

$$I - D^T D = W^T W$$

V.M. Popov

Single Constraint Quadratic Optimization

Original SDP Problem

$$\begin{array}{ll} \text{minimize} & x^T A_0 x + 2b_0^T x + c_0 \\ \text{subject to} & x^T A_1 x + 2b_1^T x + c_1 \leq 0 \end{array}$$

$x \in \mathbb{R}^n \quad A_i \in \mathbb{S}^n$

$b_i \in \mathbb{R}^n \quad c_i \in \mathbb{R}$

Not a convex optimization problem

$$\Leftrightarrow \begin{array}{ll} \text{minimize} & \text{tr}(A_0 X) + 2b_0^T x + c_0 \\ \text{subject to} & \text{tr}(A_1 X) + 2b_1^T x + c_1 \leq 0 \\ & X = xx^T \end{array}$$

$x \in \mathbb{R}^n$

$X \in \mathbb{S}^n$

A linear objective function,

A linear inequality constraint and a nonlinear equality constraint

Relaxation/ Dual problem of the SDP

$$\begin{array}{ll} \text{minimize} & \text{tr}(A_0 X) + 2b_0^T x + c_0 \\ \text{subject to} & \text{tr}(A_1 X) + 2b_1^T x + c_1 \leq 0 \\ & X \succeq xx^T \quad \Leftrightarrow \quad \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0 \end{array}$$

S-procedure (Yakubovich's S-lemma)

Definitions $\mathcal{E}_1 = \{x | x^T F_1 x + 2g_1^T x + h_1 \le 0\}$

where $F_1 \in \mathbb{S}^n,\ g_1 \in \mathbb{R}^n,\ h_1 \in \mathbb{R}$ and $\begin{bmatrix} F_1 & g_1 \\ g_1^T & h_1 \end{bmatrix} \succeq 0$

$\mathcal{E}_2 = \{x | x^T F_2 x + 2g_2^T x + h_2 \le 0\}$

where $F_2 \in \mathbb{S}^n,\ g_2 \in \mathbb{R}^n,\ h_2 \in \mathbb{R}$ and $\begin{bmatrix} F_2 & g_2 \\ g_2^T & h_2 \end{bmatrix} \succeq 0$

$\mathcal{E}_1 \subseteq \mathcal{E}_2$ if and only if

$$\exists \lambda > 0 \quad \text{s.t.} \quad \begin{bmatrix} F_2 & g_2 \\ g_2^T & h_2 \end{bmatrix} \preceq \lambda \begin{bmatrix} F_1 & g_1 \\ g_1^T & h_1 \end{bmatrix} \qquad \text{(S)}$$

Note: Sufficient Condition is clear.

If there exists $\lambda > 0$ satisfying (S), then given $x \neq 0$,

$$\begin{bmatrix} x^T & 1 \end{bmatrix} \begin{bmatrix} F_1 & g_1 \\ g_1^T & h_1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \succeq 0 \quad \Rightarrow \quad \begin{bmatrix} x^T & 1 \end{bmatrix} \begin{bmatrix} F_2 & g_2 \\ g_2^T & h_2 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \succeq 0$$

Generalized S-Procedure

We often encounter problems with constraints of the form

$$g_0(x) \geq 0 \quad \text{for all} \quad x \in \mathbb{R}^n \tag{S1}$$

satisfying $g_1(x) \geq 0, \; \cdots, \; g_m(x) \geq 0$ (S2)

where $g_0, g_1, \cdots, g_m : \mathbb{R}^n \to \mathbb{R}$.

$\Leftrightarrow$ The set-containment constraint

$$\{x \in \mathbb{R}^n | g_1(x) \geq 0, \; \cdots, \; g_m(x) \geq 0\} \subseteq \{x \in \mathbb{R}^n | g_o(x) \geq 0\}$$

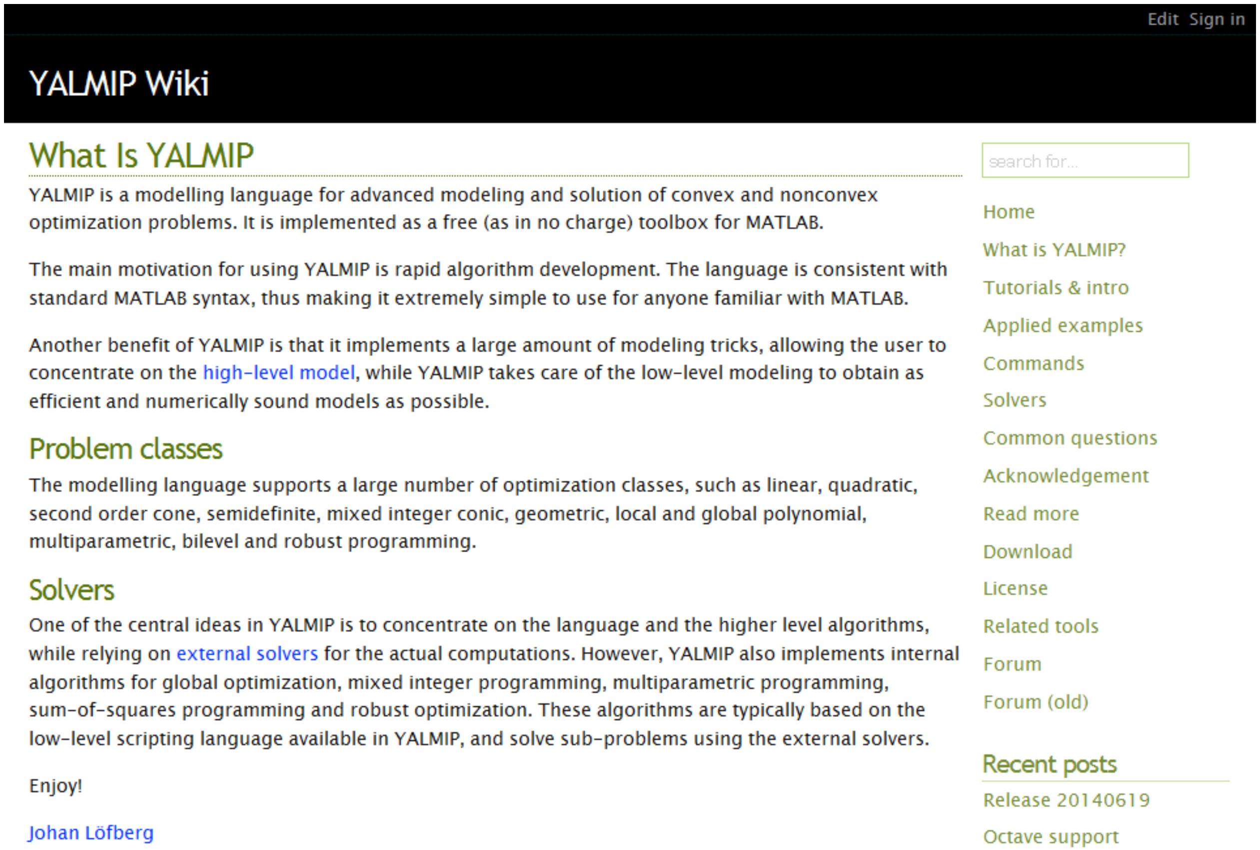
A potentially conservative but useful algebraic sufficient condition for (S1) and (S2) is the existence of positive-semidefinite functions $s_1, \cdots, s_m : \mathbb{R}^n \to \mathbb{R}$ such that

$$g_0(x) - \sum_{i=1}^{m} s_i(x) g_i(x) \geq 0 \quad \text{for all} \quad x \in \mathbb{R}^n \tag{S3}$$

For the case in which $g_0, g_1, \cdots, g_m$ are quadratic functions, the sufficient condition in (S3) is known as the S-procedure relaxation for (S1) and (S2).

LMI Programming: YALMIP

YALMIP: Yet Another LMI Parser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YALMIP Wiki

What Is YALMIP

YALMIP is a modelling language for advanced modeling and solution of convex and nonconvex optimization problems. It is implemented as a free (as in no charge) toolbox for MATLAB.

The main motivation for using YALMIP is rapid algorithm development. The language is consistent with standard MATLAB syntax, thus making it extremely simple to use for anyone familiar with MATLAB.

Another benefit of YALMIP is that it implements a large amount of modeling tricks, allowing the user to concentrate on the high-level model, while YALMIP takes care of the low-level modeling to obtain as efficient and numerically sound models as possible.

Problem classes

The modelling language supports a large number of optimization classes, such as linear, quadratic, second order cone, semidefinite, mixed integer conic, geometric, local and global polynomial, multiparametric, bilevel and robust programming.

Solvers

One of the central ideas in YALMIP is to concentrate on the language and the higher level algorithms, while relying on external solvers for the actual computations. However, YALMIP also implements internal algorithms for global optimization, mixed integer programming, multiparametric programming, sum-of-squares programming and robust optimization. These algorithms are typically based on the low-level scripting language available in YALMIP, and solve sub-problems using the external solvers.

Enjoy!

Johan Löfberg

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SDP Solvers in YALMIP

Linear Programming

(free) CDD, CLP, GLPK, LPSOLVE, QSOPT, SCIP,

(commercial) CPLEX, GUROBI, LINPROG, MOSEK, XPRESS

Mixed Integer Linear Programming

(free) CBC, GLPK, LPSOLVE, SCIP,

(commercial) CPLEX, GUROBI, MOSEK, XPRESS

Quadratic Programming

(free) BPMPD, CLP, OOQP, QPC, qpOASES, quadprogBB,

(commercial) CPLEX, GUROBI, MOSEK, NAG, QUADPROG, XPRESS

Mixed Integer Quadratic Programming

(commercial) CPLEX, GUROBI, MOSEK, XPRESS

Second-order Cone Programming

(free) ECOS, SDPT3, SeDuMi (commercial) CPLEX, GUROBI, MOSEK

Mixed Integer Second-order Cone Programming

(commercial) CPLEX, GUROBI, MOSEK

Semidefinite Programming

(free) CSDP, DSDP, LOGDETPPA, PENLAB, SDPA, SDPLR, SDPT3, SDPNAL, SeDuMi (commercial) LMILAB, MOSEK, PENBMI, PENSDP

General Nonlinear Programming and other solvers