Robust and Optimal Control, Spring 2015

Instructor: Prof. Masayuki Fujita (S5-303B)

G. Sum of Squares (SOS)

G.1 SOS Program: SOS/PSD and SDP

G.2 Duality, valid ineqalities and Cone

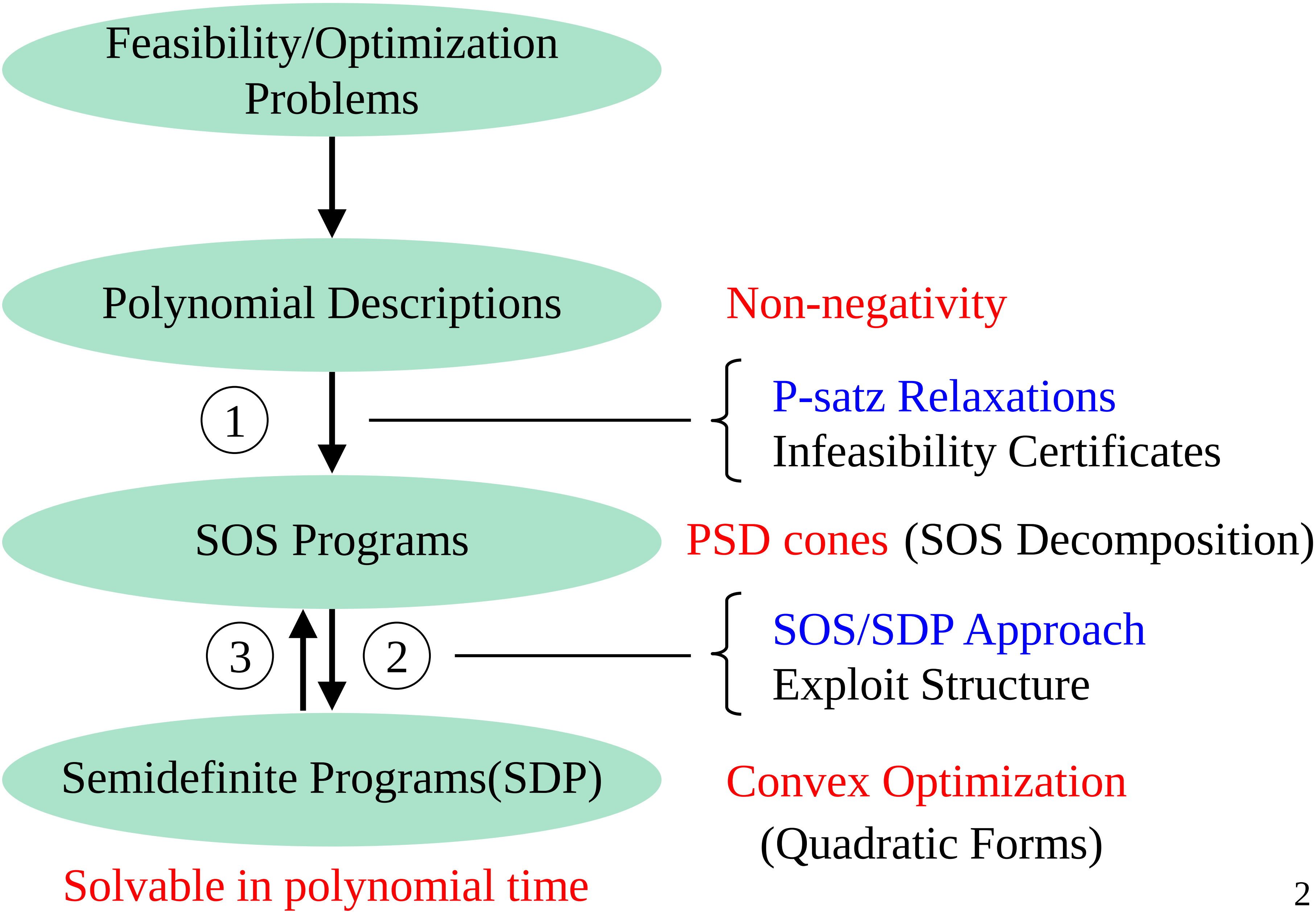
G.3 Feasibility/Optimization and Ideal

G.4 Exploiting Structure

Reference:

[BPT13] G. Blekherman, P.A. Parrilo and R.R. Thomas,
Semidefinite Optimization and Convex Algebraic Geometry,
MOS-SIAM Series on Optimization, SIAM, 2013.

Overview



Feasibility/Optimization Problems
Polynomial Descriptions
Non-negativity
1
P-satz Relaxations
Infeasibility Certificates
SOS Programs
PSD cones (SOS Decomposition)
3
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SOS/SDP Approach
Exploit Structure
Semidefinite Programs(SDP)
Convex Optimization
(Quadratic Forms)
Solvable in polynomial time

Motivating Examples

Discrete Problems: LQR with Binary Inputs

System Linear discrete-time system

$$x(t+1) = Ax(t) + Bu(t) \quad \forall t = 0, 1, \cdots, N, \; x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$$

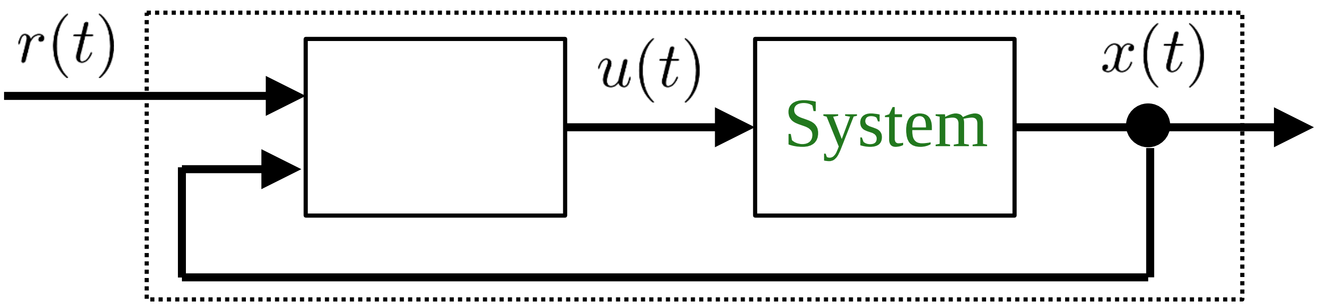
$$y(t) = x(t)$$

$$u_i(t) \in \{-1, 1\} \text{ (binary inputs) } \forall i = 1, \cdots, m \;\; \forall t = 0, \cdots, N-1$$

Objective

Given $x(0) = x_0$ and the evolution of reference signals $r(t)$,
find an optimal controller u minimizing the quadratic tracking error

$$\min_u \sum_{t=0}^{N-1} (x(t) - r(t))^T Q (x(t) - r(t))$$



Motivating Examples

Nonlinear Problems: Lyapunov Stability

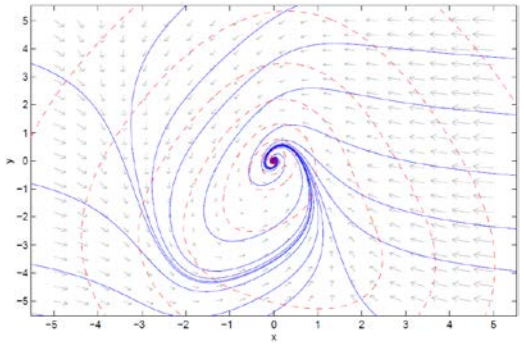
[Ex.] Moore-Greitzer model of a jet engine with controller:

$$\dot{x} = -y + 1.5x^2 - 0.5x^3$$
$$\dot{y} = 3x - y \;=: u$$

Find a Lyapunov function $V(x, y)$.

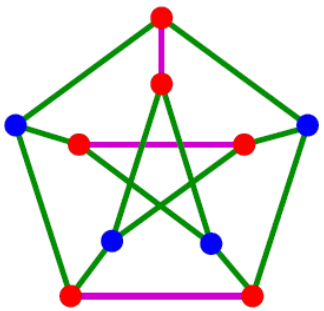
Candidate 4th order polynomial function

$$V(x,y) = \sum_{0 \le j+k \le 4} c_{jk} x^j y^k$$



Graph Problems: MAX CUT partitioning

Partition the nodes of a graph in two disjoint sets, maximizing the number of edges between sets



How to compute bounds, or exact solutions, for this kind of problems?

Polynomial Programming: Optimization Problem

minimize	$f_0(x)$	(polynomials)	
subject to	$f_i(x) \leq 0$	(polynomials)	$i = 1, \cdots, m$
	$h_i(x) = 0$	(polynomials)	$i = 1, \cdots, p$

[Ex.] Given $f \in \mathbb{R}[x_1, \cdots, x_n]$,

Primal decision problem

(P) $\exists x \in \mathbb{R}^n$ s.t. $f(x) < 0$?

↓ No

$(\neq \mathrm{P})$ $f(x) \geq 0 \quad \forall x \in \mathbb{R}^n$

f : globally non-negative

Polynomial non-negativity

(i.e., f is positive definite: PSD)

This problem is NP-hard

But, decidable.

Certificates:

Yes — Exhibit x s.t. $f(x) < 0$

No — Need a certificate/witness

i.e., a proof that
there is no feasible point

(Infeasibility certificate)

Sum of Squares(SOS) Decomposition

cf. sosdemo1.m

For $x \in \mathbb{R}^n$, a multivariate polynomial $f(x)$ is a *sum of squares* if there exist some polynomials $g_1, \cdots, g_s \in \mathbb{R}[x_1, \cdots, x_n]$ such that $f(x) = \sum_{i=1}^{s} g_i^2(x)$. Then, $f(x)$ is nonnegative.

We can write any polynomial as a quadratic function of monomials

[Ex.] $f(x,y) = 4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4$

$$= \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 4 & 2 & -\lambda \\ 2 & -7+2\lambda & -1 \\ -\lambda & -1 & 10 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} = z^T Q(\lambda) z \quad \forall \lambda \in \mathbb{R}$$

If for some λ, we have $Q(\lambda) \succeq 0$, then we can factorize $Q(\lambda)$

$$Q(6) = \begin{bmatrix} 0 & 2 \\ 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -3 \end{bmatrix} \Rightarrow f = \left\| \begin{matrix} 2xy + y^2 \\ 2x^2 + xy - 3y^2 \end{matrix} \right\|^2 = \left\| \begin{matrix} g_1(x,y) \\ g_2(x,y) \end{matrix} \right\|^2$$

$f(x,y) = z^T Q(6) z$ is an SOS decomposition $\Leftrightarrow$

(≠ P) $f(x) \geq 0 \quad \forall x \in \mathbb{R}^n$

f : Polynomial non-negativity

SOS and Semidefinite Programming (SOS/SDP)

Suppose $f \in \mathbb{R}[x_1, \cdots, x_n]$, of degree $2d$

Let z be a vector of all monomials of degree less than or equal to d

$f(x)$ is SOS iff $\exists Q$ such that $Q \succeq 0$ and $f = z^T Q z$ [SDP]

The number of components of z is ${}_{n+d}C_d = \dfrac{(n+d)!}{n!d!}$

Comparing terms gives affine constraints on the elements of Q

If Q is a feasible point of the SDP, then to construct the SOS representation

Factorize $Q = VV^T$, and write $V = [v_1 \;\; \cdots \;\; v_r]$, so that

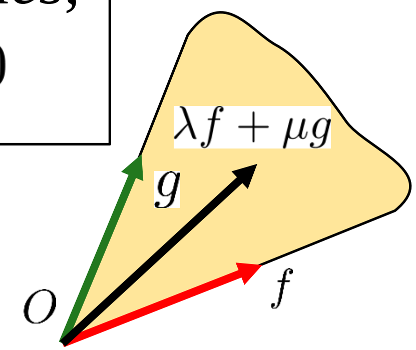
$$f = z^T V V^T z = \|V^T z\|^2 = \sum_{i=1}^{r} (v_i^T z)^2$$

One can factorize using e.g., Cholesky or eigenvalue decomposition

The number of squares r equals the rank of Q

Convexity

The set of PSD and SOS polynomials are a *convex cones*,
i.e., f, g are PSD $\Rightarrow$ $\lambda f + \mu g$ is PSD $\forall \lambda, \mu \geq 0$



Let $P_{n,d}$ be the set of SPD polynomials of degree $\leq d$
Let $\Sigma_{n,d}$ be the set of SOS polynomials of degree $\leq d$

Both $P_{n,d}$ and $\Sigma_{n,d}$ are convex cones in $\mathbb{R}^n$ where $N = {}_{n+d}C_d$

We know $\Sigma_{n,d} \subset P_{n,d}$, and testing if $f \in P_{n,d}$ is NP-hard

But testing if $f \in \Sigma_{n,d}$ is an SDP (but a learge one)

PSD = SOS iff

(i) $d = 2$: quadratic polynomials

(ii) $n = 1$: univariate polynomials

(iii) $d = 4$, $n = 2$: quartic polynomials in two variables

In general, f is PSD does not imply f is SOS

David Hilbert

⇨ Every PSD polynomial is a SOS of rational functions

Why does this work?

Three independent facts, theoretical and experimental:

1. The existence of efficient algorithms for SDP
2. The size of the SDPs grows much slower than the Bezout number μ
   - -- A bound on the number of (complex) critical points
   - -- A reasonable estimate of complexity
   - -- $\mu = (2d-1)^n$ (for dense polynomials)
   - -- Almost all (exact) algebraic techniques scale as μ
3. The lower bound f^{SOS} very often coincides with f^*
   - -- Why? What does often mean?

SOS provides short proofs, even though they are not guaranteed to exists

Help on SOS

U. Topcu, A. Packard, P. Seiler, G. Balas,
“Help on SOS,” IEEE Control Systems Magazine, 30-4, 18/23, 2010

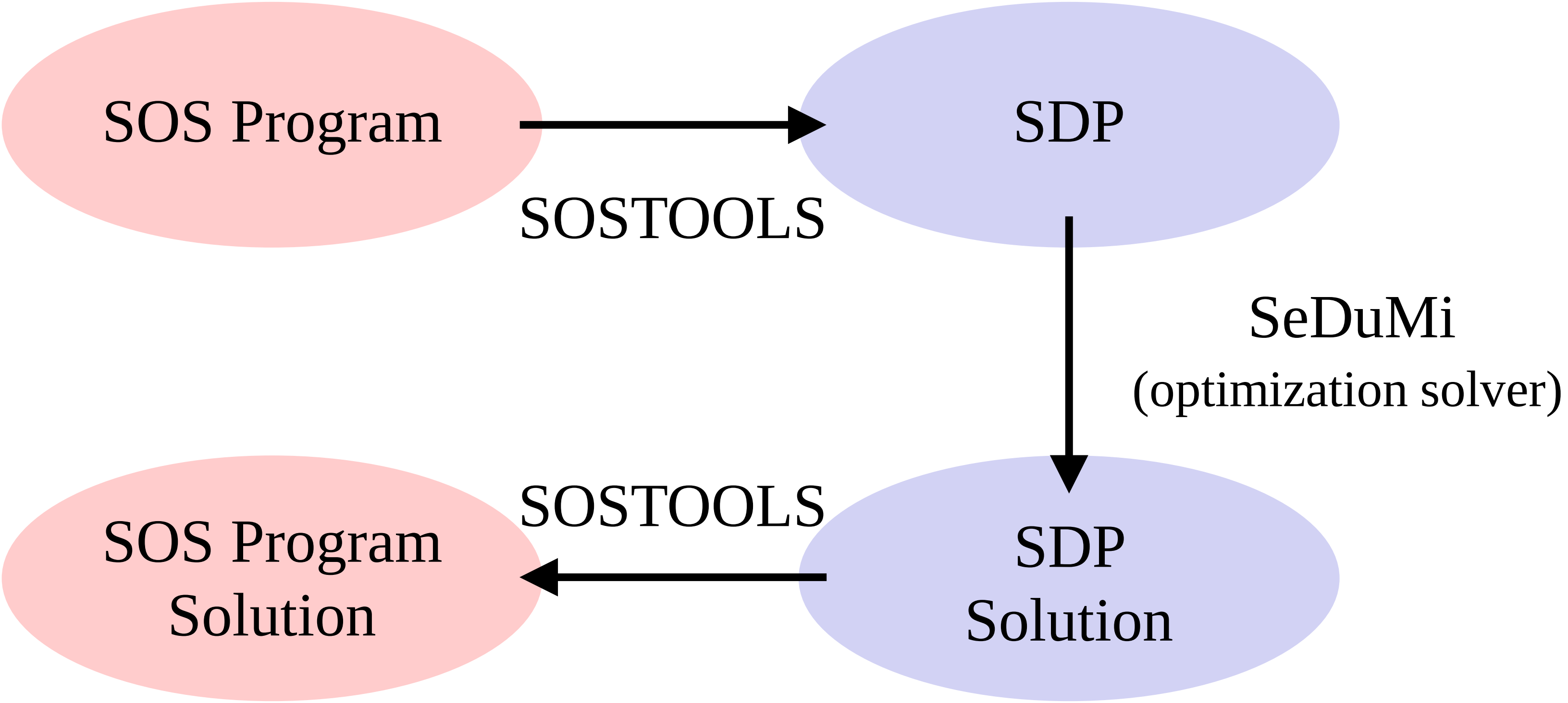
Ufuk Topcu

MATLAB Toolbox: SOS TOOLS

(with MATLAB symbolic toolbox, SeDuMi)
http://www.cds.caltech.edu/sostools/

Pablo A. Parrilo

Diagram Depicting Relations



1) Initialize a SOS Program, declare the SOS program variable
2) Define SOS program constraints
3) Set objective function (for optimization problem)
4) Call solver
5) Get solutions

SOS Problem: Example of SOSTOOLS

Problem

sosdemo2.m

Find a polynomial

$$V(x) = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2$$

(a_i : the unknown decision variables)

MATLAB Command

```
syms x1 x2 x3 ;
vars = [ x1; x2; x3 ] ;
syms a1 a2 a3 ;
decvars = [ a1; a2; a3 ] ;

f = [ -x1^3 –x1*x3^2;
    -x2 -x1^2*x2;
    -x3 -3*x3/(x3^2+1) +3*x1^2*x3 ] ;

Program1 = sosprogram(vars,decvars);
```

Constraints

$$V(x) - (x_1^2 + x_2^2 + x_3^2) \geq 0$$

$$-(x_3^2+1)\left(\frac{\partial V}{\partial x_1}\dot{x}_1 - \frac{\partial V}{\partial x_2}\dot{x}_2 - \frac{\partial V}{\partial x_3}\dot{x}_3\right) \geq 0$$

MATLAB Command

```
V = a1*x1^2 +a2*x2^2 +a3*x3^2 ;
C1 = V –( x1^2 +x2^2 +x3^2 ) ;
Program1 = sosineq( Program1, C1 ) ;

Vdot = diff(V,x1)*f(1) + diff(V,x2)*f(2) + diff(V,x3)*f(3);
C2 = -Vdot*(x3^2+1) ;
Program1 = sosineq( Program1, C2 ) ;
```

Solution

$$V(x) = 7.1525 x_1^2 + 5.7870 x_2^2 + 2.1434 x_3^2$$

MATLAB Command

```
Program1 = sossolve( Program1 ) ;
SOLV = sosgetsol( Program1, V )
```

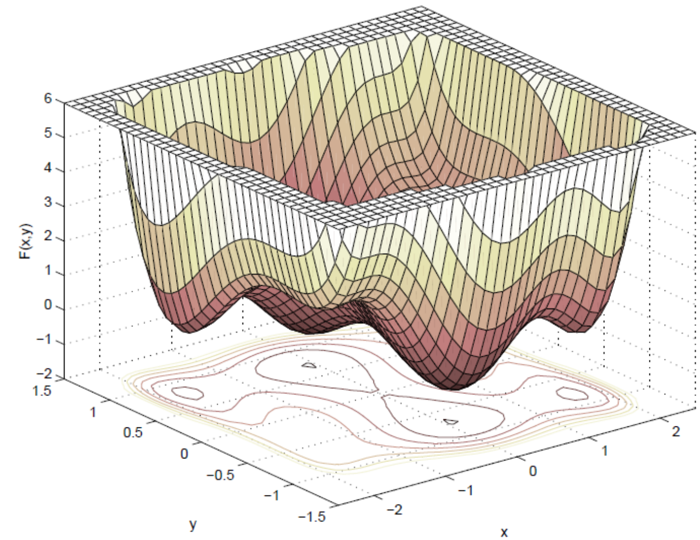
SOS program: Global optimization

cf. sosdemo3.m

Problem: $\min_{x,y} F(x,y)$

with $F(x,y) := 4x^2 - \frac{21}{10}x^4 + \frac{1}{3}x^6 + xy - 4y^2 + 4y^4$

Not convex. Many local minima. NP-hard.
How to find good lower bounds?



Find the largest γ s.t.

$F(x,y) - \gamma$ is SOS

If exact, can recover optimal solution

Surprisingly effective

Solving, the maximum γ is -1.0316. Exact bound.

P.A. Parrilo and B. Sturmfels, “Minimizing polynomial functions,” Algorithmic and quantitative real algebraic geometry, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, 60, 83/99, AMS, 2003

SOS program: Coefficient Space

Problem: Let $f_{\alpha\beta}(x) = x^4 + (\alpha + 3\beta)x^3 + 2\beta x^2 - \alpha x + 1$

What is the set of values of $(\alpha, \beta) \in \mathbb{R}^2$ for which $f_{\alpha\beta}$ is PSD? SOS?

SOS decomposition

$$f_{\alpha\beta}(x) = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^T \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$$

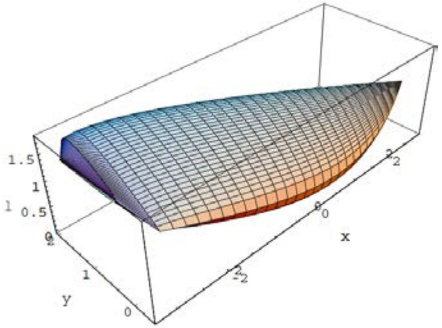
$$= q_{33}x^4 + 2q_{23}x^3 + (q_{22} + 2q_{13})x^2 + 2q_{12}x + q_{11}$$

Feasible set (satisfying PSD)

$$\left\{ (\alpha, \beta) \;\middle|\; \exists \lambda \text{ s.t. } \begin{bmatrix} 1 & -(1/2)\alpha & \beta - \lambda \\ -(1/2)\alpha & 2\lambda & (1/2)(\alpha + 3\beta) \\ \beta - \lambda & (1/2)(\alpha + 3\beta) & 1 \end{bmatrix} \succeq 0 \right\}$$

$f_{\alpha\beta}$: univariate polynomials ⇒ PSD = SOS

Convex and *semi-algebraic*



SOS program: Lyapunov Stability Analysis

To prove asymptotic stability of $\dot{x} = f(x)$ at $x = 0$

$$V(0) = 0, \dot{V}(0) = 0,$$
$$V(x) > 0, \ \dot{V}(x) = \nabla V(x) \cdot f(x) < 0 \ \forall x \neq 0$$

A. Lyapunov

[Ex.] $\dot{x} = Ax$ ⇨ $V(x) = x^T P x$ s.t. $P > 0, \ A^T P + PA < 0$

Check nonnegativity (affine of *quadratic forms*: LMI)	⇨	Check SOS conditions (affine of *polynomials*)

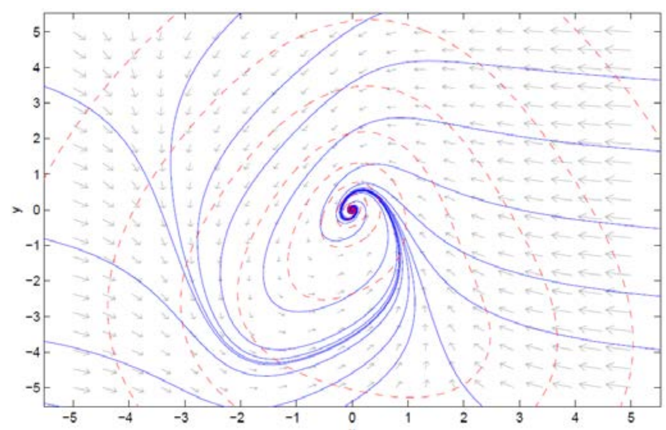
[Ex.] Moore-Greitzer model of a jet engine with controller

Find a $V(x,y)$ s.t. $V(x,y)$ is SOS and $-\dot{V}(x,y)$ is SOS

Both conditions are affine in the coefficients, so can use SOS/SDP

Resulting Lyapunov function

$$\begin{aligned} V = {} & 4.5819x^2 - 1.5786xy + 1.7834y^2 \\ & -0.12739x^3 + 2.5189x^2y - 0.34069xy^2 \\ & +0.61188y^3 + 0.47537x^4 - 0.052424x^3y \\ & +0.44289x^2y^2 + 1.8868 \cdot 10^{-6}xy^3 + 0.090723y^4 \end{aligned}$$



SOS program: Lyapunov Stability Analysis

[Ex.] Autonomous System

$$\dot{x} = -x + (1+x)y$$
$$\dot{y} = -(1+x)x$$

For the system, we can find a Lyapunov function with quartic polynomial

$$V(x,y) = 6x^2 - 2xy + 8y^2 - 2y^3 + 3x^4 + 6x^2y^2 + 3y^4$$

$$V(x,y) = \begin{bmatrix} x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 6 & -1 & 0 & 0 & 0 \\ -1 & 8 & 0 & 0 & -1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & -1 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix} \quad \text{is SOS}$$

M. Krstic

$$-\dot{V}(x,y) = \begin{bmatrix} x \\ y \\ x^2 \\ xy \end{bmatrix}^T \begin{bmatrix} 10 & 1 & -1 & 1 \\ 1 & 2 & 1 & -2 \\ -1 & 1 & 12 & 0 \\ 1 & -2 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ x^2 \\ xy \end{bmatrix} \quad \text{is SOS}$$

The matrices are positive definite, so this proves asymptotic stability

Nonlinear Control Synthesis

Lyapunov stability criterion (asymptotic stability)

For $\dot{x} = f(x)$, a Lyapunov function $V(x)$ satisfies

$$V(0) = 0\ ,\quad V(x) > 0,\quad x \neq 0 \qquad \left(\frac{\partial V}{\partial x}\right)^T f(x) < 0,\ \forall x \neq 0$$

↕ "Dual"

Anders Rantzer

a "dual" Lyapunov function: $\nabla \cdot (\rho f) > 0$

The synthesis problem is convex in $(\rho, u\rho)$: $\nabla \cdot [\rho(f + gu)] > 0$

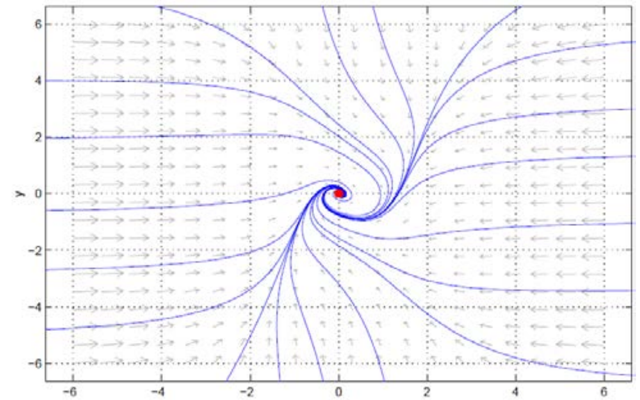
Parametrizing $(\rho, u\rho)$, can apply SOS methods

[Ex.]

$$\dot{x} = y - x^3 + x^2$$
$$\dot{y} = u$$

A stabilizing controller is:

$$u(x, y) = -1.22x - 0.57y - 0.129y^3$$



A. Rantzer, "An converse theorem for density functions," *Proc. 41st IEEE Conf. on Decision and Control*, pp. 1890-1891, 2002.

Summary: About SOS/SDP

Semi-definite matrices are SOS quadratic forms

SOS polynomials are embedded into PSD cone

$f(x)$ is SOS iff $\exists Q$ such that $Q \succeq 0$ and $f = z^T Q z$ [SDP]

The resulting SDP problem is polynomially sized (in n, for fixed d)

By properly choosing the monomials, we can exploit structure
(sparsity, symmetries, ideal structure, graph structure, etc.)

Important Feature: The problem is still a SDP
if the coefficients of F are variable, and the dependence is affine

Can optimize over SOS polynomials in affinely described families

[Ex.] If $p(x) = p_0(x) + \alpha p_1(x) + \beta p_2(x)$,
we can "easily" find values of α, β for which $p(x)$ is SOS

This fact (Exploiting this structure) will be crucial in applications

Dual Problem: Motivating Example

Primal Problem: $\min \; x_1 x_2$

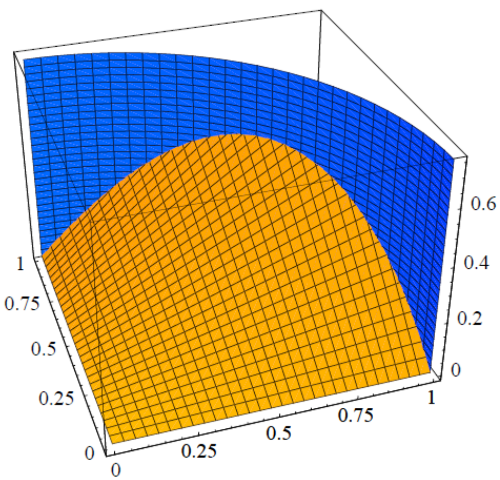
subject to $x_1 \geq 0 \quad x_2 \geq 0$

$x_1^2 + x_2^2 \leq 1$

Dual Problem: $\max \; g(\lambda)$

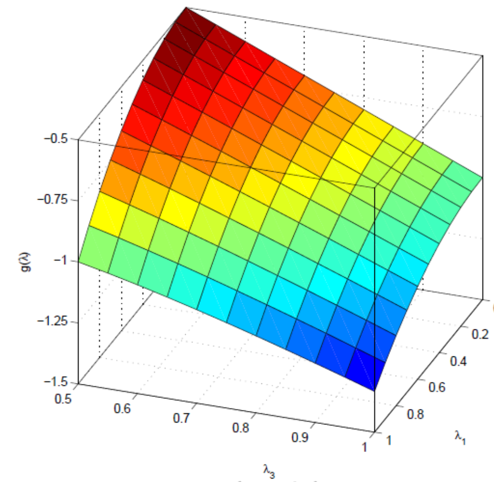
subject to $\lambda_1 \geq 0 \quad \lambda_2 \geq 0$

$\lambda_3 \geq 1/2$



Non-convex

Duality Gap



Optimal $g(\lambda^*) = -1/2$

at $\lambda^* = (0, 0, 1/2)$

Lagrange dual function:

$$g(\lambda) = \inf_x x_1 x_2 - \lambda_1 x_1 - \lambda_2 x_2 + \lambda_3 (x_1^2 + x_2^2 - 1)$$

$$= \begin{cases} -\lambda_3 - \frac{1}{2} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}^T \begin{bmatrix} 2\lambda_3 & 1 \\ 1 & 2\lambda_3 \end{bmatrix}^{-1} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} & \text{if } \lambda_3 > 1/2 \\ -\infty & \text{otherwise, except boundary condition} \end{cases}$$

The Dual is not intrinsic

The dual problem, and its corresponding optimal value, are not properties of the primal feasible set and objective function alone

(Instead, they depend on the particular equations and inequalities used)

To construct equivalent primal optimization problems with different duals:

(1) Replace the objective $f_0(x)$ by $h(f_0(x))$ where h is increasing

(2) Introduce new variables and associated constraints, e.g.,

$$\text{minimize} \quad (x_1 - x_2)^2 + (x_2 - x_3)^2$$

$$\Rightarrow \quad \begin{array}{ll} \text{minimize} & (x_1 - x_2)^2 + (x_4 - x_3)^2 \\ \text{subject to} & x_2 = x_4 \end{array}$$

(3) Add redundant constraints

Recall: Motivating Example

Primal Problem': $\min \; x_1 x_2$

subject to $x_1 \geq 0 \quad x_2 \geq 0$

$x_1^2 + x_2^2 \leq 1$

$x_1 x_2 \geq 0$

(adding the redundant constraint)

$\longleftrightarrow$

Dual Problem: $\max \; g(\lambda)$

subject to $\lambda_1 \geq 0 \quad \lambda_2 \geq 0$

$2\lambda_3 \geq 1 - \lambda_4$

$\lambda_4 \geq 0$

(The same primal feasible set and same optimal value as before)

(This problem may be written as an SDP using Schur complement)

Optimal $g(\lambda^*) = 0$

at $\lambda^* = (0, 0, 0, 1)$

Lagrange dual function:

$$g(\lambda) = \inf_x x_1 x_2 - \lambda_1 x_1 - \lambda_2 x_2 + \lambda_3(x_1^2 + x_2^2 - 1) - \lambda_4 x_1 x_2$$

$$= \begin{cases} -\lambda_3 - \frac{1}{2} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}^T \begin{bmatrix} 2\lambda_3 & 1-\lambda_4 \\ 1-\lambda_4 & 2\lambda_3 \end{bmatrix}^{-1} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} & \text{if } 2\lambda_3 > 1 - \lambda_4 \\ -\infty & \text{otherwise, except boundary condition} \end{cases}$$

Adding redundant constraints makes the dual bound tighter

This always happens! Such constraints are called valid inequalities

Algebraic Geometry

There is a correspondence
between the geometric object (the feasible subset of $\mathbb{R}^n$)
and the algebraic object (the cone of valid inequalities)

This is a dual relationship

The dual problem is constructed from the cone

For equality constraints, there is another algebraic object;
the ideal generated by the equality constraints

For optimization, we need to look both at the geometric objects
(for the primal) and the algebraic objects (for the dual problem)

An Algebraic Approach to Duality

Feasibility Problem:

? $\exists x \in \mathbb{R}^n$ s.t. $f_i(x) \geq 0$ (polynomials) $i = 1, \cdots, m$

[Ex.]

Primal Problem'':

Given $t \in \mathbb{R}$, ? $\exists x \in \mathbb{R}^2$ s.t.

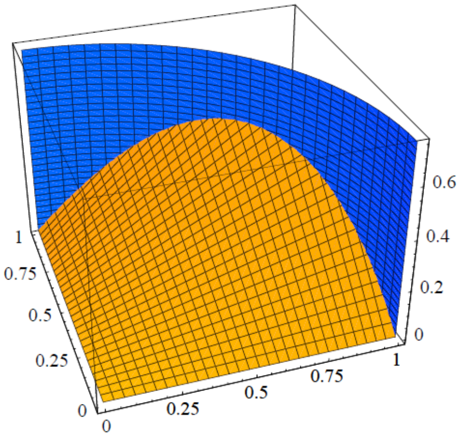
$x_1 x_2 \leq t \quad x_1 \geq 0 \quad x_2 \geq 0 \quad x_1^2 + x_2^2 \leq 1$

⟺

? $S \neq \emptyset$ where

$$S = \{x \in \mathbb{R}^n \mid f_i(x) \geq 0 \ \forall i = 1, \cdots, m\}$$

with
$$f_1(x) = t - x_1 x_2$$
$$f_2(x) = 1 - x_1^2 - x_2^2$$
$$f_3(x) = x_1$$
$$f_4(x) = x_2$$



Optimal of dual problem $g(\lambda^*) = -1/2$ at $\lambda^* = (0, 0, 1/2)$

An Algebraic Approach to Duality

Every polynomial in $\text{cone}\{f_1, \cdots, f_m\}$ is nonnegative on the feasible set

> If there is a polynomial $q \in \text{cone}\{f_1, \cdots, f_m\}$ which satisfies
> $$q(x) \le -\epsilon < 0 \qquad \forall x \in \mathbb{R}^n$$
> then the primal problem is infeasible

[Ex.] (Cont.) Let $q(x) = f_1(x) + \frac{1}{2} f_2(x)$.

Then clearly $q \in \text{cone}\{f_1, f_2, f_3, f_4\}$ and

$$q(x) = t - x_1 x_2 + \tfrac{1}{2}(1 - x_1^2 - x_2^2) = t + \tfrac{1}{2} - \tfrac{1}{2}(x_1 + x_2)^2 \le t + \tfrac{1}{2}$$

So for any $t < -\frac{1}{2}$, the primal problem is infeasible.

This corresponds to Lagrange multipliers $(1, \frac{1}{2})$ for the theorem of alternatives

(Alternatively, this is a proof by contradiction

If there exists x such that $f_i(x) \ge 0, i = 1, \cdots, 4$ then we must also have $q(x) \ge 0$, since $q \in \text{cone}\{f_1, f_2, f_3, f_4\}$

But we proved that q is negative if $t < -\frac{1}{2}$)

An Algebraic Approach to Duality

[Ex.] (Cont.) Let $q(x) = f_1(x) + f_3(x) f_4(x) = t$.

giving the stronger result that for any $t < 0$ the inequalities are infeasible. Again, this corresponds to Lagrange multipliers $(1, 1)$

In both of these examples, we found q in the cone which was globally negative. We can view q as the Lagrange function evaluated at a particular vale of λ

The Lagrange multiplier procedure is searching over a particular subset of functions in the cone; those which are generated by linear combinations of the original constraints

By searching over more functions in the cone we can do better

---

Normalization

$$q(x) = f_1(x) + \tfrac{1}{2} f_2(x) = t + \tfrac{1}{2} - \tfrac{1}{2}(x_1 + x_2)^2$$

We can also show that $-1 \in \text{cone}\{f_1, \cdots, f_4\}$, which gives a very simple proof of primal infeasibility.

$$-1 = a_0 \underbrace{q(x)}_{\text{In the cone}} + a_1 \underbrace{(x_1 + x_2)^2}_{\text{SOS}} \text{ with } a_0 = \frac{-2}{2t+1} > 0 \, , \; a_1 = \frac{-1}{2t+1} > 0$$

An Algebraic Dual Problem

Feasibility Problem:

? $\exists x \in \mathbb{R}^n$ s.t. $f_i(x) \geq 0$ (polynomials) $i = 1, \cdots, m$

↕ "Dual"

Dual Feasibility Problem:

? $-1 \in \text{cone}\{f_1, \cdots, f_m\}$

If the dual problem is feasible, then the primal problem is infeasible

In fact, a result called the Positivstellensatz(P-satz) implies that strong duality holds here

The above algebraic procedure is searching over conic combinations

An Algebraic Dual Problem

[Ex.] Linear inequalities

> **Feasibility Problem:**
>
> $? \; \exists x \in \mathbb{R}^n$ s.t. $Ax \geq 0 \quad c^T x \leq -1$

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}$$

Let us define $f_i(x) = a_i^T x$ for $i = 1, \cdots, m$ and $f_{m+1}(x) = -1 - c^T x$

Searching over the function $q(x) = \sum_{i=1}^{m} \lambda_i f_i(x) + \mu f_{m+1}(x)$

> **Dual Feasibility Problem:**
>
> $? \; \exists \lambda_i \geq 0, \;\; \mu \geq 0$ s.t. $q(x) = -1 \quad \forall x \in \mathbb{R}^n$

The above dual condition is $\lambda^T A x + \mu(-1 - c^T x) = -1 \quad \forall x \in \mathbb{R}^n$
which holds iff $A^T \lambda = \mu c$ and $\mu = 1$

> **Frakas lemma**
>
> If $\exists \lambda \in \mathbb{R}^m$ s.t. $A^T \lambda = c \; \wedge \; \lambda \geq 0$
>
> Then there does not exist $x \in \mathbb{R}^n$ s.t. $Ax \geq 0 \; \wedge \; c^T x \leq -1$

An Algebraic Dual Problem

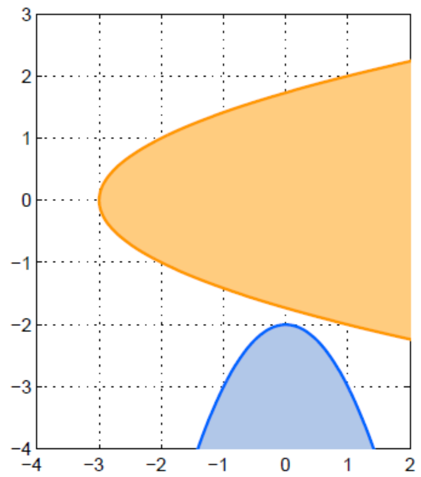
[Ex.]

Feasibility Problem:

$$S = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) \geq 0, g(x, y) \geq 0\}$$

where $f(x, y) = x - y^2 + 3$

$g(x, y) = -y - x^2 - 2$



By the P-satz, the primal is infeasible iff there exist polynomials $s_0, s_1, s_2, s_3 \in \Sigma$ such that

$$-1 = s_0 + s_1 f + s_2 g + s_3 f g$$

A certificate is given by

$$s_0 = \frac{1}{3} + 2\left(y + \frac{3}{2}\right)^2 + 6\left(x - \frac{1}{6}\right)^2, \; s_1 = 2\,, \; s_2 = 6\,, \; s_3 = 0$$

Optimization Problem

Optimization Problem:

minimize	$f_0(x)$	(polynomials)	
subject to	$f_i(x) \geq 0$	(polynomials)	$i = 1, \cdots, m$

Corresponding Feasibility Problem:

? $\exists x \in \mathbb{R}^n$ s.t.

$$t - f_0(x) \geq 0$$

$$f_i(x) \geq 0 \qquad i = 1, \cdots, m$$

Optimization Problem':

maximize t

subject to

$$t - f_0(x) + \sum_{\{i\}} s_i(x) f_i(x) + \sum_{\{i,j\}} s_{ij}(x) f_i(x) f_j(x) \leq 0$$

where $s_\alpha \in \mathbb{R}[x]$ are SOS

The variables here are (coefficients of) the polynomials $s_\alpha \in \mathbb{R}[x]$

We will see later how to approach this kind of problem using SDP

Feasibility of Semi-algebraic Set

Suppose S is a semi-algebraic set represented by polynomial inequalities and equations

$$S = \left\{ x \in \mathbb{R}^n \middle| \begin{array}{l} f_i(x) \geq 0, \ \forall i = 1, \cdots, m \\ h_j(x) = 0, \ \forall j = 1, \cdots, p \end{array} \right\}$$

Feasibility Problem: $S \neq \emptyset$?

[Non-trivial result] the feasibility problem is decidable

But NP-hard (even for a single polynomial, as we have seen)

We would like to certify infeasibility

The positivstellensatz(P-satz)

Gilbert Stengle

$$S = \emptyset \quad \Leftrightarrow \ -1 \in \mathrm{cone}\{f_1, \cdots, f_m\} + \mathrm{ideal}\{h_1, \cdots, h_p\}$$

To prove infeasibility, find $f \in \mathrm{cone}\{f_i\}, \ h \in \mathrm{ideal}\{h_i\}$ such that

$$f + h = -1$$

G. Stengle: “A Nullstellensatz and a Positivstellensatz in Semialgebraic Geometry,” Mathematische Annalen, **207** (2): 87–97, 1974

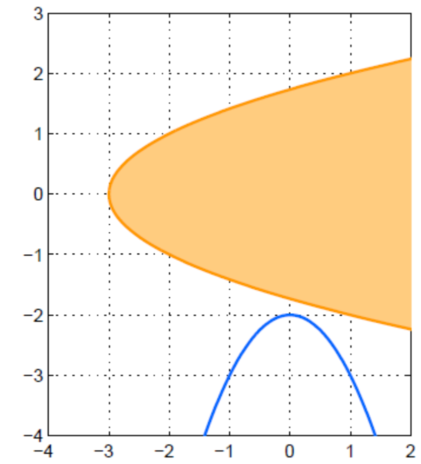
Feasibility of Semi-algebraic Set

[Ex.]

Feasibility Problem:

$$S = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) \geq 0, h(x,y) = 0\}$$

where $f(x,y) = x - y^2 + 3$
$h(x,y) = y + x^2 + 2$



By the P-satz, the primal is infeasible iff there exist polynomials $s_1, s_2 \in \Sigma$ such that

$$-1 = s_1 + s_2 f + th$$

A certificate is given by

$$s_1 = \frac{1}{3} + 2\left(y + \frac{3}{2}\right)^2 + 6\left(x - \frac{1}{6}\right)^2, \; s_2 = 2 \;, \; t = -6$$

Feasibility of Semi-algebraic Set

[Ex.] Farkas Lemma

Primal Feasibility Problem:

$? \ \exists x \in \mathbb{R}^n$ s.t. $Ax + b \geq 0 \quad Cx + d = 0$

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}$$

Let $f_i(x) = a_i^T x + b_i$, $h_i(x) = c_i^T x + d_i$

Then this system is infeasible iff

$$-1 \in \text{cone}\{f_1, \cdots, f_m\} + \text{ideal}\{h_1, \cdots, h_p\}$$

Searching over linear combinations, the primal is infeasible if there exist $\lambda \geq 0$ and μ such that

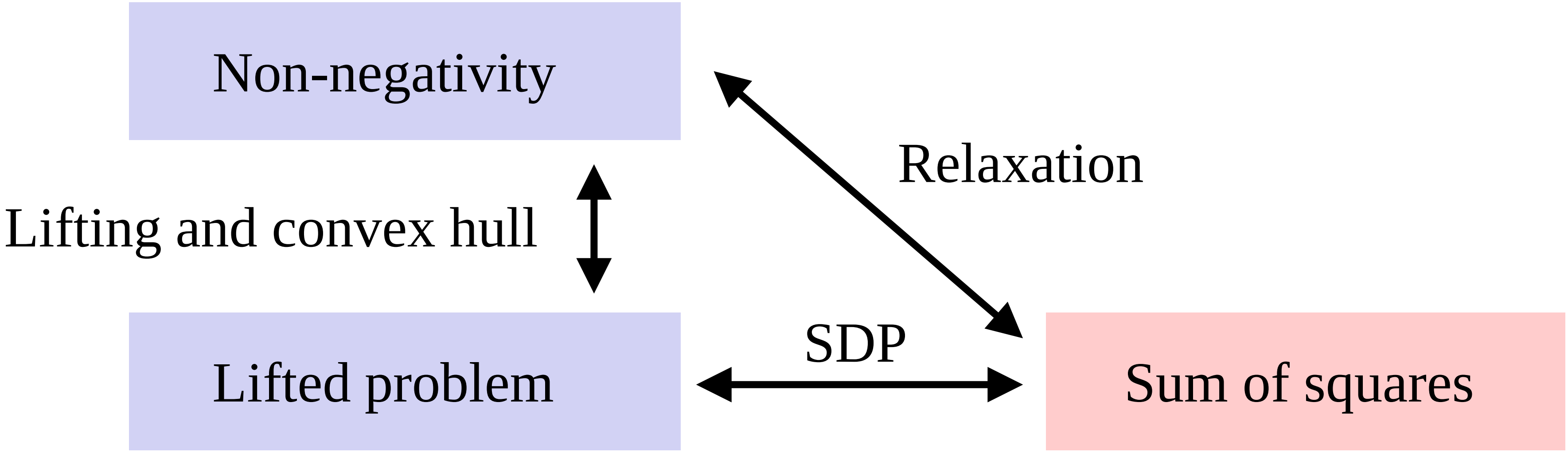
$$\lambda^T (Ax + b) + \mu^T (Cx + d) = -1$$

Equating coefficients, this is equivalent to

Primal Feasibility Problem':

$? \ \exists \lambda, \mu$ s.t. $\lambda^T A + \mu^T C = 0 \quad \lambda^T b + \mu^T d = -1 \quad \lambda \geq 0$

Relaxation scheme



Directly provides hierarchies of bounds for optimization

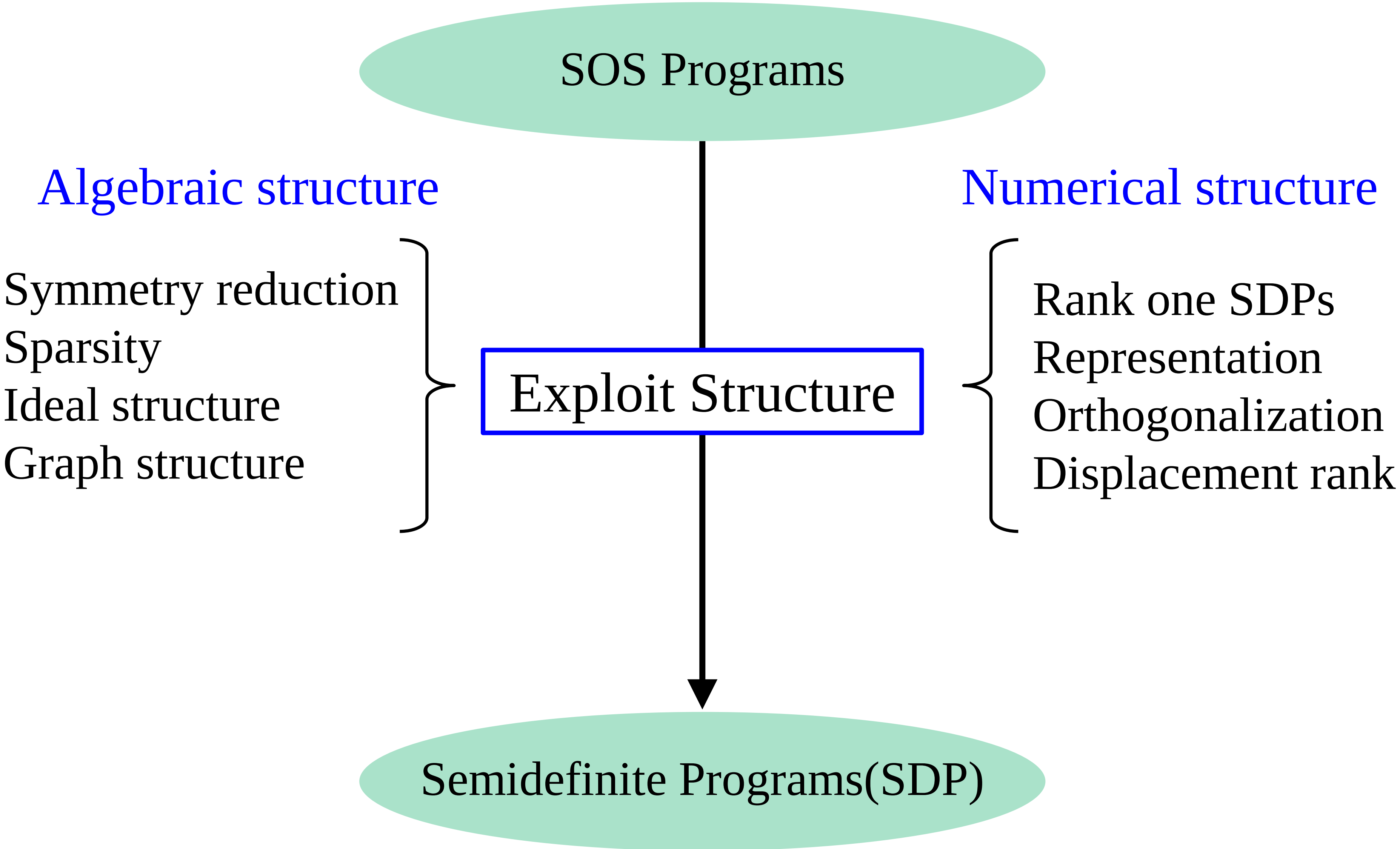
P. Parrilo, Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization, Ph.D. dissertation, *California Institute of Technology*, 2000.

Many related open questions:

- What sets have "nice" SDP representations?
- Links to "rigid convexity" and hyperbolic polynomials

However, they are a *very special* kind of SDP, with very rich algebraic and combinatorial properties

Exploiting Structure



Exploiting this structure is *crucial* in applications.

Semi-algebraic games

Game with an *infinite* number of pure strategies.

In particular, strategy sets are semi-algebraic, defined by polynomial equations and inequalities

Simplest case:
two players, zero-sum, payoff given by $P(x,y)$, strategy space is a product of intervals.

Theorem:

> The value of the game, and the corresponding optimal mixed strategies, can be computed by solving a single SOS program

Perfect generalization of the classical LP for finite games

Related results for multiplayer games and correlated equilibria

SOS Decomposition

[Ex.] $F(x,y) = 2x^4 + 5y^4 - x^2y^2 + 2x^3y$

$$= \begin{bmatrix} x^2 \\ y^2 \\ xy \end{bmatrix}^T \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} x^2 \\ y^2 \\ xy \end{bmatrix}$$

$$= q_{11}x^4 + q_{22}y^4 + (q_{33} + 2q_{12})x^2y^2 + 2q_{13}x^3y + 2q_{23}xy^3$$

F is SOS iff $\exists Q$ satisfying the SDP

$$Q \succeq 0 \qquad q_{11} = 2 \qquad q_{22} = 5 \qquad q_{33} + 2q_{12} = -1$$
$$2q_{13} = 2 \qquad 2q_{23} = 0$$

An SDP with equality constraints. Solving, we obtain:

$$Q = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix} = L^T L, \quad L = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

and therefore $F(x,y) = \frac{1}{2}(2x^2 - 3y^2 + xy)^2 + \frac{1}{2}(y^2 + 3xy)^2$

Polynomials in one variable

If $f \in \mathbb{R}[x]$, then f is SOS iff f is PSD

Every PSD scalar polynomial is the sum of one or two squares

[Ex.] $$\begin{aligned} f(x) &= x^6 - 10x^5 + 51x^4 - 166x^3 + 342x^2 - 400x + 200 \\ &= (x-2)^2(x-(2\pm i))(x-(1\pm 3i)) \\ &= (x-2)^2((x^2-3x-1) \pm i(4x-7)) \\ &= (x-2)^2((x^2-3x-1)^2 + (4x-7)^2) \end{aligned}$$

(All real roots must have even multiplicity and highest coefficient is positive)

Quadratic Polynomials

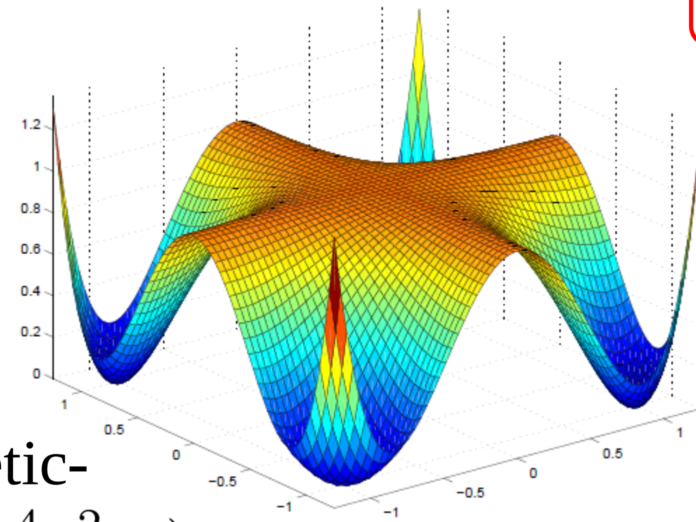
A quadratic polynomial f in n variables is PSD iff f is SOS

f is PSD iff $f(x) = x^T Q x$ where $Q \succeq 0$

f is SOS iff $f(x) = \sum_i (v_i^T x)^2 = x^T \left(\sum_i v_i v_i^T \right) x =: x^T Q x$

The Motzkin Polynomial

A positive semidefinite polynomial,
that is not a sum of squares



$$M(x,y) = x^2y^4 + x^4y^2 + 1 - 3x^2y^2$$

Nonnegativity follows from the arithmetic-geometric inequality applied to $(x^2y^4, x^4y^2, 1)$

Introduce a nonnegative factor $x^2 + y^2 + 1$

Solving the SDPs we obtain the decomposition:

$$\begin{aligned} &(x^2 + y^2 + 1)M(x,y) \\ = \; &(x^2y - y)^2 + (xy^2 - x)^2 + (x^2y^2 - 1)^2 \\ &+ \frac{1}{4}(xy^3 - x^3y)^2 + \frac{3}{4}(xy^3 + x^3y - 2xy)^2 \end{aligned}$$

The Univariate Case

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{2d} x^{2d}$$

$$= \begin{bmatrix} 1 \\ x \\ \vdots \\ x^d \end{bmatrix}^T \begin{bmatrix} q_{00} & q_{01} & \cdots & q_{0d} \\ q_{01} & q_{11} & \cdots & q_{1d} \\ \vdots & \vdots & \ddots & \vdots \\ q_{0d} & q_{1d} & \cdots & q_{dd} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ \vdots \\ x^d \end{bmatrix}$$

$$= \sum_{i=0}^{d} \left(\sum_{j+k=i} q_{jk} \right) x^i$$

In the univariate case,
the SOS condition is exactly equivalent to nonnegativity

The matrices A_i in the SDP have a Hankel structure
This can be exploited for efficient computation

Necessary Conditions

Suppose $f = c_d x^d + c_{d-1} x^{d-1} + \cdots + c_1 x + c_0$; then

f is PSD $\Rightarrow$ d is even, $c_d > 0$ and $c_0 \geq 0$

What is the analogue in n variables?

[Ex.] The Newton polytope

Suppose $f = \sum_{\alpha \in M} c_\alpha x^\alpha$

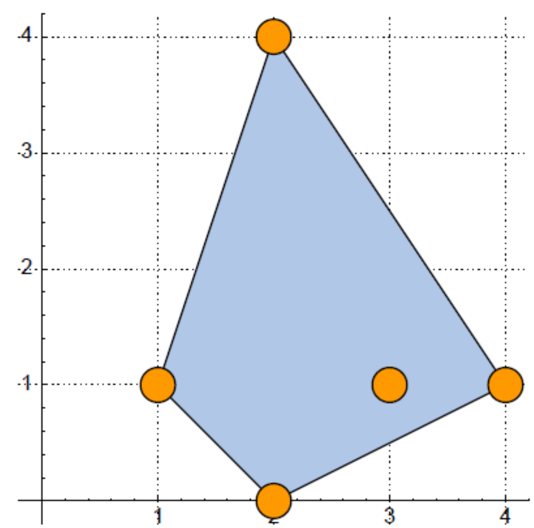
The set of monomials $M \subset \mathbb{N}^n$ is called the *frame* of f

The *Newton polytope* of f is its convex hull

$$\mathrm{new}(f) = \mathrm{co}(\mathrm{frame}(f))$$

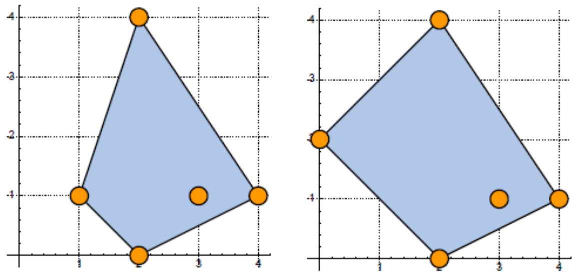
The example shows

$$f = 7x^4y + x^3y + x^2y^4 + x^2 + 3xy$$



Necessary Conditions for nonnegativity

If $f \in \mathbb{R}[x_1, \cdots, x_n]$ is PSD, then
every vertex of $\mathrm{new}(f)$ has even coordinates, and a positive coefficient



[Ex.] $f = 7x^4y + x^3y + x^2y^4 + x^2 + 3xy$

is not PSD, since term $3xy$ has coords $(1, 1)$

[Ex.] $f = 7x^4y + x^3y - x^2y^4 + x^2 + 3y^2$

is not PSD, since term $-x^2y^4$ has a negative coefficient

Properties of Newton Polytopes

Products: $\mathrm{new}(fg) = \mathrm{new}(f) + \mathrm{new}(g)$

Consequently, $\mathrm{new}(f^n) = n\mathrm{new}(f)$

If f and g are PSD polynomials then

$$f(x) \leq g(x)\ \forall x \in \mathbb{R}^n \quad \Rightarrow \quad \mathrm{new}(f) \subseteq \mathrm{new}(g)$$

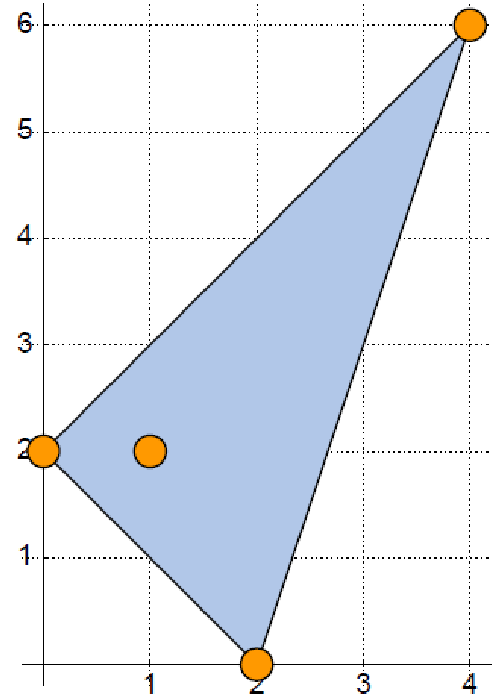
$$f = \sum_{i=1}^t g_i^2 \quad \Rightarrow \quad \mathrm{new}(g_i) \subseteq \frac{1}{2}\mathrm{new}(f)$$

SOS decomposition

Sparse SOS Decomposition

[Ex.] Find an SOS representation for

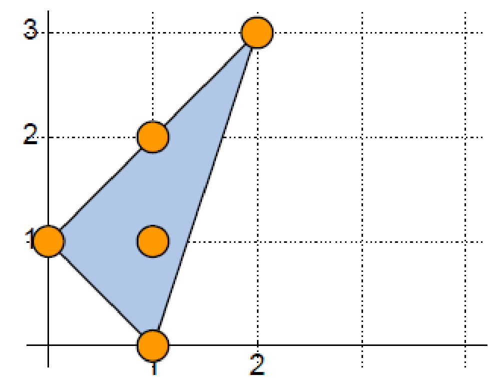
$$f = 4x^4y^6 + x^2 - xy^2 + y^2$$



The squares in an SOS decomposition can only contain the monomials

$$\frac{1}{2}\text{new}(f) \cap \mathbb{N}^n = \{x^2y^3, xy^2, xy, x, y\}$$

Without using sparsity, we would include all 21 monomials of degree less than 5 in the SDP



With sparsity, we only need 5 monomials

SOS Programming: SPOT

SPOT: Systems Polynomial Optimization Tools

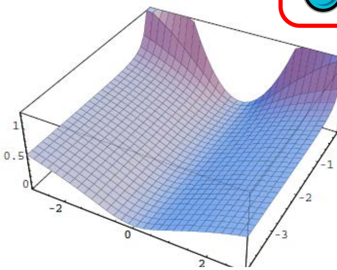
Alexandre Megretski

Software: http:// web.mit.edu/ameg/www/images/spot-20101216.zip

Manual: http://web.mit.edu/ameg/www/images/spot_manual.pdf

SOS Problems: Extensions

Other linear (partially) differential inequalities



1. Lyapunov: $V(x) \leq 0, \; \left(\frac{\partial V}{\partial x}\right)^T f(x) \leq 0, \; \forall x$

2. Hamilton-Jacobi: $V(x,t) \leq 0, \; -\frac{\partial V}{\partial t} + \mathcal{H}\left(x, \frac{\partial V}{\partial x}\right) \leq 0, \; \forall (x,u,t)$

Many possible variations:

Nonlinear H_∞ analysis, parameter dependent Lyapunov functions,
Constrained nonlinear systems, systems with time-delay, hybrid systems
mixed integer programming(MIP) problem etc.

Can also do local results (for instance, on compact domains)

Polynomial and rational vector fields, or functions with
an underlying algebraic structure

Natural extension of the LMIs for the linear case

Only for analysis, proper synthesis is a trick problem

Valid inequalities and Cones

The function $f : \mathbb{R}^n \to \mathbb{R}$ is called a valid inequality if

$$f(x) \geq 0 \qquad \text{for all feasible } x$$

Given a set of inequality constraints, we can generate others as follows:

(1) If f_1 and f_2 define valid inequalities, then so does $h(x) = f_1(x) + f_2(x)$

(2) If f_1 and f_2 define valid inequalities, then so does $h(x) = f_1(x) f_2(x)$

(3) For any f , the function $h(x) = f(x)^2$ defines a valid inequality

$\mathbb{R}[x_1, \cdots, x_n]$: the set of polynomial functions on $\mathbb{R}$ with real coefficients

A set of polynomials $P \subset \mathbb{R}[x_1, \cdots, x_n]$ is called a cone if

(1) $f_1 \in P$ and $f_2 \in P$ implies $f_1 + f_2 \in P$

(2) $f_1 \in P$ and $f_2 \in P$ implies $f_1 f_2 \in P$

(3) $f \in \mathbb{R}[x_1, \cdots, x_n]$ implies $f^2 \in P$

It is called a proper cone if $-1 \notin P$

By applying the above rules to the inequality constraint functions (algebra), we can generate a cone of valid inequalities

Algebra: Cones

For $S \subset \mathbb{R}^n$, the cone defined by S is

$$\mathcal{C}(S) = \{f \in \mathbb{R}[x_1, \cdots, x_n] | f(x) \geq 0 \ \forall x \in S\}$$

If P_1 and P_2 are cones, then so is $P_1 \cap P_2$

Every cone contains the set of SOS polynomials Σ, which is the smallest cone

The set $\mathrm{monoid}\{f_1, \cdots, f_m\} \subset \mathbb{R}[x_1, \cdots, x_n]$ is the set of all finite products of polynomials f_i, together with 1

The smallest cone containing the polynomials $f_1, \cdots, f_m$ is

$$\mathrm{cone}\{f_1, \cdots, f_m\}$$
$$= \{\sum_{i=1}^{r} s_i g_i \mid s_0, \cdots, s_r \in \Sigma, \ g_i \in \mathrm{monoid}\{f_1, \cdots, f_m\}\}$$

If $f_1, \cdots, f_m$ are valid inequalities, then so is every polynomial in $\mathrm{cone}\{f_i\}$

The polynomial h is an element of $\mathrm{cone}\{f_1, \cdots, f_m\}$ iff

$$h(x) = s_0 + \sum_{\{i\}} s_i g_i + \sum_{\{i,j\}} s_{ij} g_i g_j + \sum_{\{i,j,k\}} s_{ijk} g_i g_j g_k + \cdots$$

where a coefficient $s_\alpha \in \mathbb{R}[x]$ that is a sum of squares

(linear combination of squarefree products of f_i with SOS coefficients)

An algebraic Dual Problem: Interpretation

Searching the Core

Lagrange duality is searching over linear combinations with nonnegative coefficients to fine a globally negative function as a certificate

$$h(x) = \lambda_1 f_1 + \cdots + \lambda_m f_m$$

The algebraic procedure is searching over conic combinations

$$h(x) = s_0 + \sum_{\{i\}} s_i g_i + \sum_{\{i,j\}} s_{ij} g_i g_j + \sum_{\{i,j,k\}} s_{ijk} g_i g_j g_k + \cdots$$

where a coefficient $s_\alpha \in \mathbb{R}[x]$ that is a sum of squares

Formal Proof

View $f_1, \cdots, f_m$ are predicates, with $f_i(x) \geq 0$ meaning that x satisfies f_i

Then $\mathrm{cone}\{f_1, \cdots, f_m\}$ consists of predicates which are logical consequences of $f_1, \cdots, f_m$

If we find -1 in the cone, then we have a proof by contradiction

The objective is to automatically search the cone for negative functions: i.e., proofs of infeasibility

Frakas Lemma

(Algebraic definition)

Frakas lemma states that the following are strong alternatives:

(i)	$\exists \lambda \in \mathbb{R}^m$ s.t. $A^T \lambda = c \ \wedge \lambda \geq 0$
(ii)	$\exists x \in \mathbb{R}^n$ s.t. $Ax \geq 0 \ \wedge c^T x < 0$

(Geometric interpretation) ($\because$ Lagrangian duality)

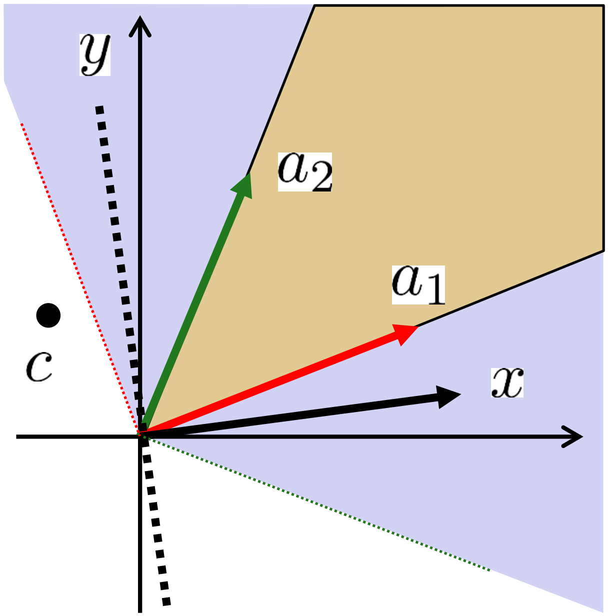
(i) c is in the convex cone

$$\{A^T \lambda \mid \lambda \geq 0\}$$

(ii) x defines the hyperplane

$$\{y \in \mathbb{R}^n \mid y^T x = 0\}$$

with separates c from the cone



Valid Equality constraints and Ideals

The function $f : \mathbb{R}^n \to \mathbb{R}$ is called a valid equality constraint if

$$f(x)=0 \qquad \text{for all feasible } x$$

Given a set of equality constraints, we can generate others as follows:

(1) If f_1 and f_2 are valid equalities, then so is $f_1 + f_2$

(2) For any $h \in \mathbb{R}[x_1, \cdots, x_n]$, if f is a valid equality, then so is hf

Using these will make the dual bound tighter

A set of polynomials $I \subset \mathbb{R}[x_1, \cdots, x_n]$ is called a ideal if

(1) $f_1 + f_2 \in I$ for all $f_1 \in I$, $f_2 \in I$

(2) $hf \in I$ for all $f \in I$ and $h \in \mathbb{R}[x_1, \cdots, x_n]$

Given $f_1, \cdots, f_m$, we can generate an ideal of valid equalities by repeatedly applying these rules

Algebra: Ideals

This gives the ideal generated by $f_1, \cdots, f_m$,

$$\text{ideal}\{f_1, \cdots, f_m\} = \left\{ \sum_{i=1}^{m} h_i f_i \mid h_i \in \mathbb{R}[x_1, \cdots, x_n] \right\}$$

Generator of an ideal

Every polynomial in $\text{ideal}\{f_1, \cdots, f_m\}$ is a valid equality

$\text{ideal}\{f_1, \cdots, f_m\}$ is the smallest ideal containing $f_1, \cdots, f_m$

The polynomials $f_1, \cdots, f_m$ are called the generators/basis of the ideal

Properties of ideals

If I_1 and I_2 are ideals, then so is $I_1 \cap I_2$

An ideal generated by one polynomial is called a principal ideal

Algebra: The real Nullstellensatz(N-satz)

Σ : the cone of polynomials representable as SOS

Suppose $h_1, \cdots, h_m \in \mathbb{R}[x_1, \cdots, x_n]$

$$-1 \in \Sigma + \text{ideal}\{h_1, \cdots, h_m\} \;\Leftrightarrow\; \mathcal{V}_{\mathbb{R}}\{h_1, \cdots, h_m\} = \emptyset$$

Equivalently, there is no $x \in \mathbb{R}^n$ such that

$$h_i(x) = 0 \; \forall i = 1, \cdots, m$$

iff there exists $t_1, \cdots, t_m \in \mathbb{R}[x_1, \cdots, x_n]$ and $s \in \Sigma$ such that

$$-1 = s + t_1 h_1 + \cdots + t_m h_m$$

[Ex.] Suppose $h(x) = x^2 + 1$ Then clearly $\mathcal{V}_{\mathbb{R}}\{h\} = \emptyset$

We saw earlier that the complex N-satz cannot be used to prove

But we have $-1 = s + th$ with $s(x) = x^2$ and $t(x) = -1$

and so the real N-satz implies $\mathcal{V}_{\mathbb{R}}\{h\} = \emptyset$

The polynomial equation $-1 = s + th$ gives a certificate of infeasibility

Algebra: The Positivstellensatz(P-satz)

Feasibility Problem for basic semi-algebraic sets:

$$? \ \exists x \in \mathbb{R}^n \ \text{ s.t. } \quad \begin{array}{ll} f_i(x) \geq 0 & i = 1, \cdots, m \\ h_j(x) = 0 & j = 1, \cdots, p \end{array}$$

Call the feasible set S ; recall

Every polynomial in $\text{cone}\{f_1, \cdots, f_m\}$ is nonnegative on S

Every polynomial in $\text{ideal}\{h_1, \cdots, h_p\}$ is zero on S

$$S = \emptyset \quad \Leftrightarrow \quad -1 \in \text{cone}\{f_1, \cdots, f_m\} + \text{ideal}\{h_1, \cdots, h_p\}$$

Centerpiece of real algebraic geometry

Dual Feasibility Problem:

? $\exists t_i \in \mathbb{R}[x_1, \cdots, x_n]$ and $\exists s_\alpha \in \Sigma$ s.t.

$$-1 = \sum_{\{i\}} t_i h_i + s_0 + \sum_{\{i\}} s_i f_i + \sum_{\{i,j\}} s_{ij} f_i f_j + \cdots$$

These are strong alternatives

Testing the P-satz

Dual Feasibility Problem:

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$$-1 = \sum_{\{i\}} t_i h_i + s_0 + \sum_{\{i\}} s_i f_i + \sum_{\{i,j\}} s_{ij} f_i f_j + \cdots$$

This is a convex feasibility problem in t_i, s_α

To solve it, we need to choose a subset of the cone to search; i.e., the maximum degree of the above polynomial; then the problem is a SDP

This gives a hierarchy of syntactically verifiable certificates

The validity of a certificate may be easily checked;
e.g., linear algebra, random sampling

Unless NP=co-NP, the certificates cannot always be polynomially sized

Infeasibility Certificates

and associated computational techniques

	Complex numbers	Real numbers
Linear	Range/Kernel Linear algebra	Farkas lemma Linear programming
Polynomial	N-satz Bounded degree: linear algebra, Groebner bases	Polynomial systems over $\mathbb{R}$ P-satz (a central result in real algebraic geometry) Bounded degree: SDP SOS are a fundamental ingredient

Common generalization of Hilbert's N-satz and LP duality

Guarantees the existence of infeasibility certificates for real solutions of systems of polynomial equations

Infeasibility Certificates: Some Theorems

(Range/Kernel)

$Ax = b$ is infeasible

$$\Updownarrow$$

$\exists \mu$ s.t. $A^T\mu = 0, \; b^T\mu = -1$

(Farkas Lemma)

$$\begin{aligned} Ax + b &= 0 \\ Cx + d &\geq 0 \end{aligned}$$ is infeasible

$$\Updownarrow$$

$\exists \lambda \geq 0, \; \mu$ s.t. $$\begin{aligned} A^T\mu + C^T\lambda &= 0 \\ b^T\mu + d^T\lambda &= -1 \end{aligned}$$

(Hilbert's N-satz)

Let $f_i(z), i = 1, \cdots, m$ be polynomials in $z \in \mathbb{C}^n$. Then,

$f_i(z) = 0, \; i = 1, \cdots, m$

is infeasible in $\mathbb{C}^n$

$$\Updownarrow$$

$-1 \in \text{ideal}(f_1, \cdots, f_m)$

(P-satz)

$$\begin{aligned} f_i(z) = 0, \; i = 1, \cdots, m \\ g_i(z) \geq 0, \; i = 1, \cdots, p \end{aligned}$$

is infeasible in $\mathbb{R}^n$

$$\Updownarrow$$

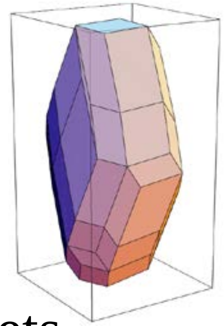
$\exists F(x), G(x) \in \mathbb{R}[x]$ s.t.

$$\begin{aligned} F(x) + G(x) &= -1 \\ F(x) &\in \text{ideal}(f_1, \cdots, f_m) \\ G(x) &\in \text{cone}(g_1, \cdots, g_p) \end{aligned}$$

Liftings

By going to higher dimensional representations, things may become easier:

"complicated" sets can be the projection of much simpler ones



A polyhedron in $\mathbb{R}^n$ with a "small" number of faces can project to a lower dimensional space with exponentially many faces

Basic semialgebraic sets can project into non-basic semialgebraic sets

An essential technique in integer programming

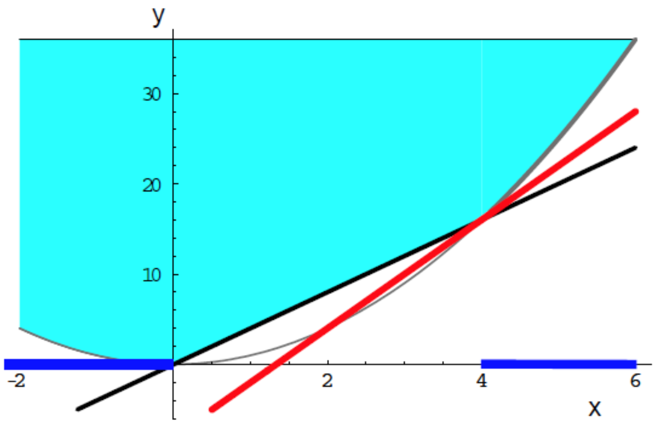
[Ex.] minimize $(x-3)^2$ subject to $x(x-4) \geq 0$

Feasible set $[-\infty, 0] \cup [4, \infty]$ Not convex

Lifting $L(x) = (x, x^2) =: (x, y)$

minimize $y - 6x + 9$

subject to $y - 4x \geq 0$ and $\begin{bmatrix} 1 & x \\ x & y \end{bmatrix} \succeq 0$



The dual side of SOS: Moment sequaences

The SDP dual of the SOS construction gives efficient semidefinite liftings

[Ex.] For the univariate case

$$L : \mathbb{R} \to \mathbb{S}^{d+1} \quad \text{with} \quad L(x) = \begin{bmatrix} 1 & x & \cdots & x^d \\ x & x^2 & \cdots & x^{d+1} \\ \vdots & \vdots & \ddots & \vdots \\ x^d & x^{d+1} & \cdots & x^{2d} \end{bmatrix}$$

The matrices $L(x)$ are Hankel, positive semidefinite, and rank one

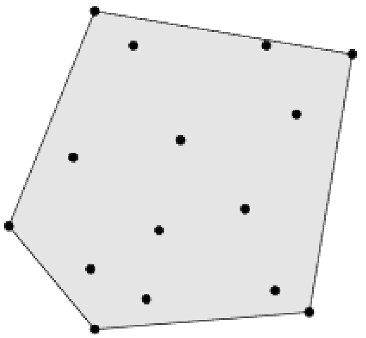
The convex hull $\mathrm{conv} L(x)$ contains only PSD Hankel matrices

$$\mathrm{Hankel}(w) := \begin{bmatrix} 1 & w_1 & \cdots & w_d \\ w_1 & w_2 & \cdots & w_{d+1} \\ \vdots & \vdots & \ddots & \vdots \\ w_d & w_{d+1} & \cdots & w_{2d} \end{bmatrix}$$

In fact, in the univariate case every PSD Hankel is in the convex hull

Convex Combination, Convex Hull and Convex Cone

Convex combination of $x_1, \cdots, x_k$:

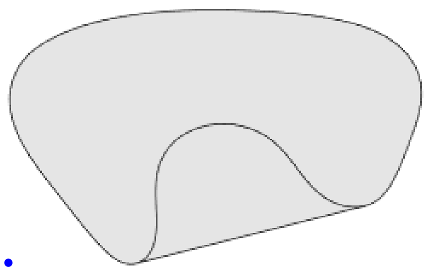


Any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

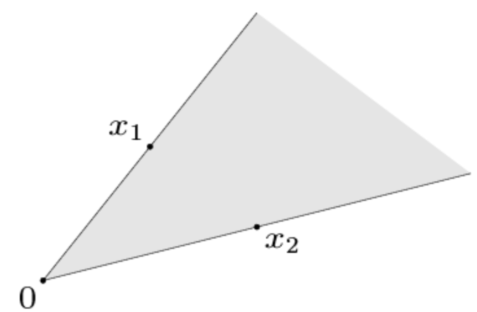
with $\theta_1 + \cdots + \theta_k = 1,\ \theta_i \geq 0$

Convex hull $\mathrm{conv}S$:



Set of all convex combinations of points in S

Conic (nonnegative) combination of x_1 and x_2 :



Any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

with $\theta_1 \geq 0,\ \theta_2 \geq 0$

Convex Cone:

Set that contains all conic combinations of points in the set

Positive Semidefinite (PSD) Cone

Notations:

$\mathcal{S}^n$: set of symmetric $n \times n$ matrices

$\mathcal{S}^n_+ = \{X \in \mathcal{S}^n | X \succeq 0\}$

: positive semidefinite $n \times n$ matrices (convex cone)

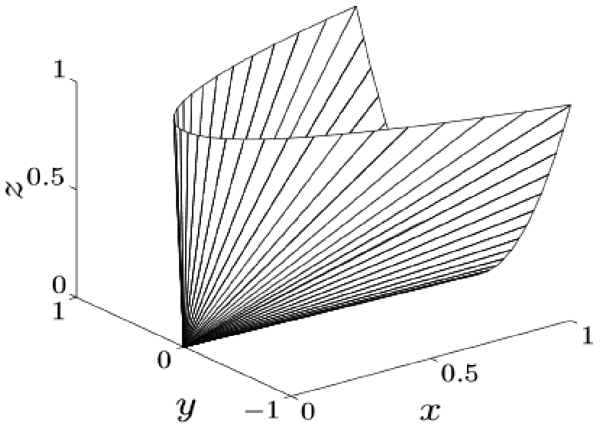
$X \in \mathcal{S}^n_+ \;\Leftrightarrow\; z^T X z \geq 0 \;\forall z$

$\mathcal{S}^n_{++} = \{X \in \mathcal{S}^n | X \succ 0\}$

: positive definite $n \times n$ matrices

[Ex.]

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathcal{S}^2_+$$



[BV04] S. Boyd and L. Vandenberghe: Convex Optimization, Cambridge University Press, 2004

Algebraic structure

Sparseness: few non-zero coefficients

Newton polytopes techniques

Ideal structure: equality constraints

SOS on *quotient rings*. Compute in the coordinate ring. Quotient bases.

Graph structure:

Dependency graph among the variables

Symmetries: invariance under a group

SOS on *invariant rings*.
Representation theory and invariant-theoretic methods

Numerical structure

Rank one SDPs

Dual coordinate change makes all constraints rank one
Efficient computation of Hessians and gradients

Representations Interpolation representation. Orthogonalization

Displacement rank Fast solvers for search direction

Algebraic Structure: SOS over everything…

Algebraic tools are *essential* to exploit problem structure:

Standard	**Equality constraints**	**Symmetries**
Polynomial ring $\mathbb{R}[x]$	Quotient ring $\mathbb{R}[x]/I$	Invariant ring $\mathbb{R}[x]^G$
Monomials $(\deg \leq k)$	*Standard* monomials	Isotypic components
$\frac{1}{(1-\lambda)^n}$	Hibert series	Molien series
$= \sum_{k=0}^{\infty} \left(\begin{array}{c} n+k-1 \\ k \end{array} \right) \cdot \lambda^k$	Finite convergence for zero dimensional ideals	Block diagonalization